TMS165/MSA350 Stochastic Calculus

Written Exam Tuesday 27 October 2015 8.30–12.30 am

Teacher and Jour: Patrik Albin, telephone 070 6945709.

Aids: Two sheets (=four pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed).

Grades: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

Motivations: All answers/solutions must be motivated.

Througout this exam $B = \{B(t)\}_{t\geq 0}$ denotes a Brownian motion. And Good Luck!

Task 1. Explain how one can actually construct a unit mean exponentially distributed random variable $X : \Omega \to \mathbb{R}$ on a sample space Ω with a probability measure $\mathbf{P}$ (such that $\mathbf{P}\{X \leq x\} = 1 - \mathrm{e}^{-x}$ for $x \geq 0$). **(5 points)**

Task 2. Show that $\int_0^t B(u)\,du - \frac{1}{3}B(t)^3$ is a martingale. **(5 points)**

Task 3. State and prove the isometry property of the Itô integral of simple adapted processes. **(5 points)**

Task 4. Find $X(t)$ if $d(\mathrm{e}^{B(t)}) = \mathrm{e}^{B(t)}\,dX(t)$ and $X(0) = 0$. **(5 points)**

Task 5. Given real numbers σ, μ and r, find the solution $f(x,t)$ to the PDE

$$\frac{\partial f(x,t)}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 f(x,t)}{\partial x^2} + \mu\frac{\partial f(x,t)}{\partial x} = rf(x,t) \quad \text{for } t\in[0,T], \quad f(x,T) = x^2.$$ **(5 points)**

Task 6. The explicit Euler method for finding a numerical solution $\hat{X}(t)$ to the SDE

$$dX(t) = \mu(X(t),t)\,dt + \sigma(X(t),t)\,dB(t) \quad \text{for } t \in (0,T], \quad X(0) = x_0,$$

based on iteration over the grid $0 = t_0 < t_1 < \ldots < t_n = T$ goes like $\hat{X}(t_0) = x_0$ and

$$\hat{X}(t_i) - \hat{X}(t_{i-1}) = \mu(\hat{X}(t_{i-1}),t_{i-1})\,(t_i - t_{i-1}) + \sigma(\hat{X}(t_{i-1}),t_{i-1})\,(B(t_i) - B(t_{i-1}))$$

for $i = 1,\ldots,n$. The fully implicit Euler method for the same task uses $\mu(\hat{X}(t_i),t_i)$ and $\sigma(\hat{X}(t_i),t_i)$ instead of $\mu(\hat{X}(t_{i-1}),t_{i-1})$ and $\sigma(\hat{X}(t_{i-1}),t_{i-1})$ above (and is typically much more "stable"): Explain why the implicit method is not just as simple as

$$\hat{X}(t_i) - \hat{X}(t_{i-1}) = \mu(\hat{X}(t_i),t_i)\,(t_i - t_{i-1}) + \sigma(\hat{X}(t_i),t_i)\,(B(t_i) - B(t_{i-1})) \quad \text{for } i = 1,\ldots,n,$$

but requires more modifications than just replacing $(\hat{X}(t_{i-1}),t_{i-1})$ with $(\hat{X}(t_i),t_i)$ at two locations. **(5 points)**

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Solutions to Written Exam 27 October 2015

Task 1. Take $\Omega = [0,\infty)$, $X(\omega) = \omega$ for $\omega \in \Omega$ and $\mathbf{P}\{[a,b]\} = \int_a^b \mathrm{e}^{-y}\,dy$ for $[a,b] \subseteq \Omega$ to obtain $\mathbf{P}\{X \leq x\} = \mathbf{P}\{\omega \in \Omega : X(\omega) \leq x\} = \mathbf{P}\{\omega \leq x\} = \mathbf{P}\{[0,x]\} = 1 - \mathrm{e}^{-x}$.

Task 2. By Itô's formula we have $d\big(\int_0^t B(u)\,du - \frac{1}{3}B(t)^3\big) = B(t)\,dt - B(t)^2\,dB(t) - B(t)\,dt = -B(t)^2\,dB(t)$, so that $\int_0^t B(u)\,du - \frac{1}{3}B(t)^3 = -\int_0^t B(u)^2\,dB(u)$ which is a martingale since $-B^2 \in E_T$ for any $T \geq 0$.

Task 3. This is Property 4 of the properties of the Itô integral of simple adapted processes listed on pages 93-94 in Klebaner's book: See the proof of that property on pages 94-95 in Klebaner's book.

Task 4. As $X(t)$ is the stochastic logarithm of $\mathrm{e}^{B(t)}$, we have $X(t) = \ln\big(\mathrm{e}^{B(t)}/\mathrm{e}^{B(0)}\big) + \int_0^t d(\mathrm{e}^{B(s)})^2/(2\,(\mathrm{e}^{B(s)})^2) = B(t) + \int_0^t ds/2 = B(t) + t/2$.

Task 5. Feynman-Kac formula gives $f(x,t) = \mathbf{E}\big\{\mathrm{e}^{-r(T-t)}X(T)^2 \big| X(t) = x\big\}$, where $X(t)$ solves the SDE $dX(t) = \mu\,dt + \sigma\,dB(t)$, so that $X(t) = X(0) + \mu\,t + \sigma\,B(t)$ and $X(T) = X(t) + \mu\,(T-t) + \sigma\,(B(T) - B(t))$, giving $f(x,t) = \mathrm{e}^{-r(T-t)}\big(\sigma^2(T-t) + (\mu\,(T-t) + x)^2\big)$.

Task 6. The reason is that $\sum_{i=1}^n \big(\sigma(X(t_i),t_i) - \sigma(X(t_{i-1}),t_{i-1})\big)\big(B(t_i) - B(t_{i-1})\big) \rightarrow [\sigma(X(t),t), B(t)] = \int_0^t d[\sigma(X(s),s), B(s)] = \int_0^t d\sigma(X(s),s)\,dB(s) = \int_0^t \big(\sigma_x'(X(s),s)\,dX(s) + \frac{1}{2}\sigma_{xx}''(X(s),s)\,d[X,X](s) + \sigma_t'(X(s),s)\,ds\big)\,dB(s) = \int_0^t \sigma_x'(X(s),s)\sigma(X(s),s)\,dB(s)^2 = \int_0^t \sigma_x'(X(s),s)\sigma(X(s),s)\,ds$ is not zero (in general).