TMS165/MSA350 Stochastic Calculus

Written Exam Friday 19 August 2016 8.30–12.30 am

TEACHER AND JOUR: Patrik Albin, telephone 070 6945709.

AIDS: Two sheets (=four pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed).

GRADES: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Througout this exam $B = \{B(t)\}_{t\geq 0}$ denotes Brownian motion.

Task 1. Express the value of the Itô integral $\int_0^t B(s)^{n-1}\,dB(s)$ in terms of just $B(t)$ and ordinary (non-Itô) integrals for an integer $n \geq 2$. **(5 points)**

Task 2. Prove that Itô integral processes of simple adapted processes are martingales. **(5 points)**

Task 3. Is it true that the stochastic logaritm of the stochastic exponential of an Itô process $X(t)$ equals $X(t)$? **(5 points)**

Task 4. The stochastic exponential $X(t)$ of Brownian motion with drift is given by

$$dX(t) = \mu\, X(t)\, dt + \sigma\, X(t)\, dB(t), \quad X(0) = 1,$$

for some constants $\mu \in \mathbb{R}$ and $\sigma > 0$. Does this diffusion process $X(t)$ have a stationary distribution? **(5 points)**

Task 5. Let $X(t) = B(t) + \sin(t)$. Find a probability measure $\mathbf{Q}$ that makes $X(t)$ a Brownian motion. **(5 points)**

Task 6. Derive Milstein's method for strong numerical solution of a time homogeneous SDE. **(5 points)**

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Solutions to Written Exam 19 August 2016

Task 1. As Itô's formula gives $d(B(t)^n) = n\, B(t)^{n-1}\, dB(t) + \frac{1}{2}\, n\, (n-1)\, B(t)^{n-2}\, dt$, we get $\int_0^t B(s)^{n-1}\, dB(s) = \frac{1}{n}\, B(t)^n - \frac{n-1}{2} \int_0^t B(s)^{n-2}\, ds$.

Task 2. This is a solved exercise in the exercise material of the course: See the solution there.

Task 3. As $Y(t) = \mathcal{E}(X)(t)$ is given by $dY(t) = Y(t)\, dX(t)$ and $Y(0) = 1$, while $Z(t) = \mathcal{L}(Y)(t)$ is given by $dY(t) = Y(t)\, dZ(t)$ and $Z(0) = 0$, we see that $dY(t) = Y(t)\, dX(t) = Y(t)\, dZ(t)$, so that $dX(t) = dZ(t)$ (as $Y(t)$ is non-zero), giving $X(t) - X(0) = Z(t) - Z(0) = Z(t) = \mathcal{L}(\mathcal{E}(X))(t)$.

Task 4. Paying respect to that $X(t)$ is a process with values in the interval $(0, \infty)$, for $X(t)$ to have a stationary distribution it is necessary that

$$\int_0^\infty \frac{1}{\sigma(x)^2}\, \exp\biggl\{\int_1^x \frac{2\, \mu(y)}{\sigma(y)^2}\, dy\biggr\}\, dx < \infty,$$

where $\mu(x) = \mu\, x$ and $\sigma(x) = \sigma\, x$. However, the intergrand of the above integral evaluates to

$$\frac{1}{\sigma^2\, x^2}\, \exp\biggl\{\int_1^x \frac{2\, \mu}{\sigma^2\, y}\, dy\biggr\} = \frac{x^{2\mu/\sigma^2}}{\sigma^2\, x^2},$$

which is not integrable over $(0, \infty)$ for any choice of $\mu \in \mathbb{R}$ and $\sigma > 0$.

Task 5. This is Exercise 10.4 in Klebaner's book: See his solution.

Task 6. See Section 2.4 in Stig's lecture notes.