TMS165/MSA350 Stochastic Calculus

Written Exam Tuesday 24 October 2017 8.30–12.30

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AIDS: Two sheets (=four pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed).

GRADES: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Througout this exam $B = \{B(t)\}_{t\geq 0}$ denotes a Brownian motion.

Task 1. Find the quadratic variation process $[X,X](t)$ of the process $X(t) = \int_0^t (t-s)\,dB(s)$. **(5 points)**

Task 2. Find coefficient functions $\mu(x)$ and $\sigma(x)$ such that the solution to the SDE $dX(t) = \mu(X(t))\,dt + \sigma(X(t))\,dB(t)$ satisfies $X(t) = \mathrm{e}^{\int_0^t (X(s)-1/2)\,ds + B(t)}$. **(5 points)**

Task 3. Find the transition probability density $p(y,t,x,s) = \frac{d}{dy}\,\mathbf{P}\{X(t)\leq y\,|\,X(s)=x\}$ when $X(t) = B(t)+t$ is Brownian motion with unit drift coefficient. **(5 points)**

Task 4. What conditions have to be required for a smooth function $f:\mathbb{R}\to\mathbb{R}$ to ensure that $f(B(t)+t)$ is a martingale? **(5 points)**

Task 5. Let $X(t) = B(t) + \sin(t)$ for $B(t)$ a $\mathbf{P}$-Brownian motion. Find $d\mathbf{Q}/d\mathbf{P}$ for an equivalent probability measure $\mathbf{Q}$ to $\mathbf{P}$ such that $X(t)_{t\in[0,T]}$ is a $\mathbf{Q}$-Brownian motion. **(5 points)**

Task 6. Grönwall's Lemma states that if $A, B > 0$ are constants and $\phi : [0,T] \to \mathbb{R}$ a continuous function such that $\phi(t) \leq A + B\int_0^t \phi(s)\,ds$ for $t \in [0,T]$ then it holds that $\phi(t) \leq A\,\mathrm{e}^{Bt}$ for $t \in [0,T]$. Prove this result. **(5 points)**

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Solutions to Written Exam 24 October 2017

Task 1. $dX(t) = d(t\,B(t)) - d(\int_0^t s\,dB(s)) = B(t)\,dt + t\,dB(t) - t\,dB(t) = B(t)\,dt$ making $d[X,X](t) = 0$.

Task 2. $dX(t) = X(t)\,\big((X(t)–1/2)\,dt + dB(t) + (1/2)\,(dB(t))^2\big) = X(t)^2\,dt + X(t)\,dB(t)$ so that $\mu(x) = x^2$ and $\sigma(x) = x$.

Task 3. $p(y,t,x,s) = \frac{d}{dy}\,\mathbf{P}\{B(t) \leq y-t\,|\,B(s) = x-s\} = \frac{d}{dy}\,\mathbf{P}\{B(t)-B(s) \leq (y-x)-(t-s)\} = f_{\mathrm{N}(0,t-s)}((y-x)-(t-s)) = \frac{1}{\sqrt{2\pi(t-s)}}\,\mathrm{e}^{((y-x)-(t-s))^2/(2(t-s))}$.

Task 4. $df(B(t)+t) = f'(B(t)+t)\,(dB(t)+dt) + \frac{1}{2}\,f''(B(t)+t)\,dt$ must have zero drift coefficient requiring that $\frac{1}{2}\,f''(x) + f'(x) = 0$ with solution $f(x) = C_1 + C_2\,\mathrm{e}^{-2x}$ for some constants $C_1, C_2 \in \mathbb{R}$. We must also check that $\mathbf{E}\{|f(B(t)+t)|\} < \infty$ but that is clearly satisfied for this choice of f.

Task 5. Noting that $X(t) = B(t) + \int_0^t \cos(s)\,ds$ Girsanov's Theorem gives $d\mathbf{Q}/d\mathbf{P} = \mathrm{e}^{-\int_0^T \cos(s)\,dB(s) - \frac{1}{2}\int_0^T \cos(s)^2\,ds}$.

Task 6. See page 2 in Stig Larsson's lecture notes.