TMS165/MSA350 Stochastic Calculus

Written Exam Thursday 4 January 2018 8.30–12.30 am

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AIDS: Two sheets (=four pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed).

GRADES: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Througout this exam $B = \{B(t)\}_{t\geq 0}$ denotes a Brownian motion.

Task 1. Explain why the Riemann-Stieltjes integral $\int f\, dg$ is not well-defined when $[f,g] \neq 0$. **(5 points)**

Task 2. Show that the quadratic covariation between two independent Brownian motions $B_1(t)$ and $B_2(t)$ is zero. **(5 points)**

Task 3. Solve the SDE

$$dX(t) = \frac{1}{2} X(t)(\ln(X(t)))^2\, dt + X(t)\ln(X(t))\, dB(t) \quad \text{for } t > 0, \quad X(0) = \mathrm{e}.$$

(5 points)

Task 4. Is $\{\int_0^t B(r)\, dr\}_{t\geq 0}$ a martingale wrt. the filtration $\mathcal{F}_t = \mathcal{F}_t^B$ generated by B? **(5 points)**

Task 5. Give the maximum likelihood estimator for the parameter μ based on observation of the solution $\{X(t)\}_{t\in[0,T]}$ of the SDE

$$dX(t) = \mu\, X(t)\, dt + \sigma\, X(t)\, dB(t), \quad X(0) = 1.$$

(5 points)

Task 6. Given sufficiently nice functions $\mu(x)$ and $\sigma(x)$ together with a constant $\lambda \in (0,1)$ a composite Euler scheme for calculation of a numerical approximation Y_N of the exact value $X(T)$ of the solution of the SDE $dX(t) = \mu(X(t))\, dt + \sigma(X(t))\, dB(t)$ for $t \in (0,T]$, $X(0) = X_0$, at $t = T$ is given by $Y_0 = X_0$ together with the recursive scheme

$$Y_n = Y_{n-1} + \big(\mu(Y_n) - f(Y_n)\big)(t_n - t_{n-1}) + \big(\lambda\sigma(Y_n) + (1-\lambda)\sigma(Y_{n-1})\big)(B(t_n) - B(t_{n-1}))$$

for $n = 1, \ldots, N$, where $0 = t_0 < t_1 < \ldots < t_N = T$. Find the function $f(x)$ that makes Y_N converge to $X(T)$ as $\max_{1\leq n\leq N} t_n - t_{n-1} \downarrow 0$. **(5 points)**

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Solutions to Written Exam 4 January 2018

Task 1. When $\int_0^T f\,dg$ is well-defined the limits $\lim_{\max_{1\leq i\leq n} t_i-t_{i-1}\downarrow 0}\sum_{i=1}^n f(t_i)\,(g(t_i)-g(t_{i-1}))$ and $\lim_{\max_{1\leq i\leq n} t_i-t_{i-1}\downarrow 0}\sum_{i=1}^n f(t_{i-1})\,(g(t_i)-g(t_{i-1}))$ must coincide for grids $0=t_0<t_1<\ldots<t_{n-1}<t_n=T$ that become infinitely fine in the limit. However, when $[f,g](T)\neq 0$ these limits do not coincide as their difference is precisely $[f,g](T)$.

Task 2. Picking a partition $0=t_0<t_1<\ldots<t_n=T$ of the interval $[0,T]$ we have $\mathbf{E}\big\{\sum_{i=1}^n(B_1(t_i)-B_1(t_{i-1}))\,(B_2(t_i)-B_2(t_{i-1}))\big\}=0$ and $\mathbf{Var}\big\{\sum_{i=1}^n(B_1(t_i)-B_1(t_{i-1}))\times(B_2(t_i)-B_2(t_{i-1}))\big\}=\sum_{i=1}^n\mathbf{Var}\{(B_1(t_i)-B_1(t_{i-1}))\,(B_2(t_i)-B_2(t_{i-1}))\}=\sum_{i=1}^n\mathbf{E}\big\{(B_1(t_i)-B_1(t_{i-1}))^2\,(B_2(t_i)-B_2(t_{i-1}))^2\big\}=\sum_{i=1}^n(t_i-t_{i-1})^2\leq T\times\max_{1\leq i\leq n}(t_i-t_{i-1})\to 0$ as $\max_{1\leq i\leq n}(t_i-t_{i-1})\downarrow 0$ so that $\sum_{i=1}^n(B_1(t_i)-B_1(t_{i-1}))\,(B_2(t_i)-B_2(t_{i-1}))\to_{\mathbb{L}^2}0$ making $[B_1,B_2]([0,T])=0$.

Task 3. Taking $Y(t)=\ln(X(t))$ Itô's formula shows that

$$\begin{aligned}dY(t)&=\frac{dX(t)}{X(t)}-\frac{d[X,X](t)}{2\,X(t)^2}\\&=\frac{1}{2}(\ln(X(t)))^2\,dt+\ln(X(t))\,dB(t)-\frac{1}{2}(\ln(X(t)))^2\,dt\\&=Y(t)\,dB(t)\end{aligned}$$

with $Y(0)=1$, so that $Y(t)=\mathrm{e}^{B(t)-t/2}$ and $X(t)=\exp\{\mathrm{e}^{B(t)-t/2}\}$.

Task 4. As $\mathbf{E}\{\int_0^t B(r)\,dr\,|\,\mathcal{F}_s\}=\int_0^s B(r)\,dr+\mathbf{E}\{\int_s^t(B(r)-B(s))\,dr\}+(t-s)\,B(s)=\int_0^s B(r)\,dr+(t-s)\,B(s)$ we see that $\int_0^t B(r)\,dr$ is not a martingale.

Task 5. By insertion in Eq. 10.52 in Klebaner's book with $\mu_1(x,t)=0$, $\mu_2(x,t)=\mu\,x$ and $\sigma(x,t)=\sigma\,x$ we find that the likelihood is

$$\Lambda=\exp\bigg\{\int_0^T\frac{\mu\,X(t)}{\sigma^2X(t)^2}\,dX(t)-\frac{1}{2}\int_0^T\frac{\mu^2X(t)^2}{\sigma^2X(t)^2}\,dt\bigg\}.$$

By solving the equation $\partial\Lambda/\partial\mu=0$ for μ we obtain the maximum likelihood estimator $\frac{1}{T}\int_0^T X(t)^{-1}\,dX(t)$ of μ.

Task 6. According to the Euler method we have the convergence

$$\sum_{n=1}^N\big(\mu(Y_{n-1})\,(t_n-t_{n-1})+\sigma(Y_{n-1})\,(B(t_n)-B(t_{n-1}))\big)\to X(T)$$

as $\max_{1\leq n\leq N}t_n-t_{n-1}\downarrow 0$. Here we have

$$\mu(Y_{n-1})\,(t_n - t_{n-1}) \approx \big(\mu(Y_n) - \mu'(Y_n)\,(Y_n - Y_{n-1})\big)\,(t_n - t_{n-1}) \approx \mu(Y_n)\,(t_n - t_{n-1})$$

and

$$\begin{aligned}\sigma(Y_{n-1})\,(B(t_n) - B(t_{n-1})) &\approx \big(\sigma(Y_n) - \sigma'(Y_n)\,(Y_n - Y_{n-1})\big)\,(B(t_n) - B(t_{n-1})) \\ &\approx \sigma(Y_n)\,(B(t_n) - B(t_{n-1})) - \sigma'(Y_n)\sigma(Y_n)\,(B(t_n) - B(t_{n-1}))^2 \\ &\approx \sigma(Y_n)\,(B(t_n) - B(t_{n-1})) - \sigma'(Y_n)\sigma(Y_n)\,(t_n - t_{n-1}),\end{aligned}$$

so that we must have $f(x) = \lambda\,\sigma'(x)\sigma(x)$.