TMS165/MSA350 Stochastic Calculus Part I

Written exam Wednesday 11 April 2012 8.30 am - 12.30 am

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Aids: None.

Grades: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

Motivations: All answers/solutions must be motivated.

Througout this exam $B = \{B(t)\}_{t\geq 0}$ is a Brownian motion. And Good Luck to you all!

Task 1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a bounded function that has continuous partial derivatives of all orders. State conditions that are necessary and sufficient for $\{f(B(t)^2, t)\}_{t\geq 0}$ to be a martingale wrt. the filtration $\{\mathcal{F}_t^B\}_{t\geq}$ generated by B. **(5 points)**

Task 2. Calculate the covariation process $\{[\sin(B(t)), \cos(B(t))]\}_{t\geq 0}$. **(5 points)**

Task 3. Consider the stochastic logarithm of the stochastic exponential $\{\mathcal{L}(\mathcal{E}(X(t)))\}_{t\geq 0}$ of an Itô process $\{X(t)\}_{t\geq 0}$. Is it true or not that $\mathcal{L}(\mathcal{E}(X(t))) = X(t)$? (The answer must be supported by a full motivation!) **(5 points)**

Task 4. Show that the stochastic process $\{W(t)\}_{t\geq 0}$ given by $W(0) = 0$ and $W(t) = t\, B(1/t)$ for $t > 0$ is a Brownian motion. **(5 points)**

Task 5. Let X be a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ that has a so called Laplace distribution under the probability measure $\mathbf{P}$, which is to say that X has probability density function $f_X(x) = \frac{1}{2}\,\mathrm{e}^{-|x|}$ for $x \in \mathbb{R}$. Find a new probability measure $\mathbf{Q}$ on the sample space and σ-field $(\Omega, \mathcal{F})$ such that X is standard normal $\mathrm{N}(0,1)$-distributed under $\mathbf{Q}$. **(5 points)**

Task 6. Consider a so called fully implicit numerical scheme given by

$$Y_0 = 1 \quad \text{and} \quad Y_k = Y_{k-1} - Y_k\,(t_k - t_{k-1}) + Y_k\,(B(t_k) - B(t_{k-1})) \quad \text{for} \;\; k = \{1, \ldots, n\}.$$

As the grid $0 = t_0 < t_1 < \ldots < t_n = T$ becomes finer and finer, what SDE will the above scheme become an approximative numerical solution of? **(5 points)**

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Solutions to written exam Wednesday 11 April 2012

Task 1. Writing $F(x,t) = f(x^2,t)$ we have $df(B(t)^2,t) = dF(B(t),t) = F'_x(B(t),t)\, dB(t) + \frac{1}{2}\, F''_{xx}(B(t),t)\, dt + F'_t(B(t),t)\, dt$, so that $f(B(t)^2,t)$ is a martingale iff. $\frac{1}{2}\, F''_{xx}(x,t) + F'_t(x,t) = 2\, x^2\, f''_{xx}(x^2,t) + f'_x(x^2,t) + f'_t(x^2,t) = 0$.

Task 2. Since $d[\sin(B(t)), \cos(B(t))] = d(\sin(B(t)))\, d(\cos(B(t))) = \big(\cos(B(t))\, dB(t) - \frac{1}{2}\, \sin(B(t))\, dt\big)\,\big(-\sin(B(t))\, dB(t) - \frac{1}{2}\, \cos(B(t))\, dt\big) = -\cos(B(t))\sin(B(t))\, dt$ we get $[\sin(B(t)), \cos(B(t))] = -\int_0^t \sin(B(s))\cos(B(s))\, ds$.

Task 3. As $\mathcal{E}(X(t)) = \mathrm{e}^{X(t)-X(0)-\frac{1}{2}[X(t),X(t)]}$ [so that in particular $\mathcal{E}(X(0)) = 1$] and $\mathcal{L}(U(t)) = \log(U(t)/U(0)) + \frac{1}{2}\int_0^t U(s)^{-2}\, d[U(s),U(s)]$, we get $\mathcal{L}(\mathcal{E}(X(t))) = X(t) - X(0) - \frac{1}{2}[X(t),X(t)] + \frac{1}{2}\int_0^t \mathcal{E}(X(s))^{-2}\,(\mathcal{E}(X(s))\, dX(s))^2 = X(t) - X(0)$, so the answer is yes if $X(0) = 0$ but no otherwise.

Task 4. As $W(t)$ clearly is a zero-mean Gaussian process it is enough to check that it has the same covariance function $\mathbf{E}\{B(s), B(t)\} = \min(s,t)$ as has Brownian motion (cf. Theorem 3.3 in Klebaner's book). However, $\mathbf{E}\{W(s), W(t)\} = \mathbf{E}\{s\, B(1/s), t\, B(1/t)\} = st\, \min(1/s, 1/t) = \min(s,t)$.

Task 5. $\mathbf{Q}(A) = \int_A 2\, \mathrm{e}^{|X|}\, \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-X^2/2}\, d\mathbf{P}$ for $A \in \mathcal{F}$.

Task 6. The scheme gives an approximate solution to the SDE $dY(t) = -Y(t)\, dt + Y(t)\, dB(t) + dY(t)\, dB(t) = -Y(t)\, dt + Y(t)\, dB(t) + \big(-Y(t)\, dt + Y(t)\, dB(t) + dY(t)\, dB(t)\big)\, dB(t) = Y(t)\, dB(t)$ for $t \in [0,T]$ with initial value $Y(0) = 1$, that is to say, an approximation of the stochastic exponential of B.