# TMS165/MSA350 Stochastic Calculus

# Written Exam Tuesday 25 October 2016 8.30–12.30 am

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AIDS: Two sheets (=four pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed).

GRADES: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Througout this exam $B = \{B(t)\}_{t\geq 0}$ denotes a Brownian motion.

**Task 1.** Solve the Stratonovich SDE

$$dX(t) = X(t)\,\partial B(t) \quad \text{for } t \geq 0, \quad X(0) = 1. \qquad \textbf{(5 points)}$$

**Task 2.** Let $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-y^2/2}\,dy$ denote the standard normal cummulative distribution function. Show that $\{\Phi(B(t)/\sqrt{T-t}\,)\}_{t\in[0,T)}$ is a martingale. **(5 points)**

**Task 3.** Find the solution $f(x,t)$ to the PDE

$$\frac{\partial f(x,t)}{\partial t} + \frac{1}{2}\,\frac{\partial^2 f(x,t)}{\partial x^2} - x\,\frac{\partial f(x,t)}{\partial x} = 0 \quad \text{for } t \in [0,T], \quad f(x,T) = x^2. \qquad \textbf{(5 points)}$$

**Task 4.** Let $X(t)$ be the solution to a time homogeneous SDE

$$dX(t) = \mu(X(t))\,dt + \sigma(X(t))\,dB(t) \quad \text{for } t \geq 0, \quad X(0) = x_0.$$

Find coefficients $\mu, \sigma : \mathbb{R} \to \mathbb{R}$ such that $\{X(t)^2 - \int_0^t X(s)^2\,ds - t\}_{t\geq 0}$ is a martingale. (All Itô integrals may be assumed to be martingales.) **(5 points)**

**Task 5.** Let $X$ be a random variable that is $\mathrm{N}(\mu_1, \sigma_1^2)$- and $\mathrm{N}(\mu_2, \sigma_2^2)$-distributed under two equivalent probability measures $\mathbf{P}_1$ and $\mathbf{P}_2$, respectively. Show that $\mathbf{P} = (\mathbf{P}_1 + \mathbf{P}_2)/2$ [i.e., $\mathbf{P}(A) = \mathbf{P}_1(A)/2 + \mathbf{P}_2(A)/2$ for all $A$] is a probability measure and that $\mathbf{P}$ is equivalent to both $\mathbf{P}_1$ and $\mathbf{P}_2$. Also, find the probability $\mathbf{P}\{X \leq 0\}$. **(5 points)**

**Task 6.** Let $X(t)$ be the solution to an SDE

$$dX(t) = \mu(X(t),t)\,dt + \sigma(X(t),t)\,dB(t) \quad \text{for } t \in [0,T], \quad X(0) = x_0,$$

with sufficiently smooth coefficients $\mu, \sigma : \mathbb{R}\times[0,T] \to \mathbb{R}$ and let $g : \mathbb{R} \to \mathbb{R}$ be a sufficiently smooth function. Describe how to obtain a numerical approximation of the expected value $\mathbf{E}\{g(X(T))\}$ and also what numerical errors that will occur for this approximation and how big they will be. **(5 points)**

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# Solutions to Written Exam 25 October 2016

**Task 1.** According to Theorem 5.20 in Klebaner $X(t)$ satisfies the Itô SDE

$$dX(t) = \frac{1}{2}\,X(t)\,dt + X(t)\,dB(t) \quad \text{for } t>0, \quad X(0)=1,$$

so that $X(t)$ is the stochastic exponential of $\frac{1}{2}\,t+B(t)$ which in turn simply is $\mathrm{e}^{B(t)}$ (see e.g., Theorem 5.2 in Klebaner). Hence we have $X(t) = \mathrm{e}^{B(t)}$.

**Task 2.** We use Itô's formula from Theorem 4.18 in Klebaner according to which for a smooth function $f(x,t)$ and a solution $X(t)$ to an SDE

$$dX(t) = \mu(X(t),t)\,dt + \sigma(X(t),t)\,dB(t) \quad \text{for } t\geq 0, \quad X(0)=x_0,$$

we have

$$df(X(t),t) = f'_x(X(t),t)\,dX(t) + \tfrac{1}{2}\,\sigma(X(t),t)^2\,f''_{xx}(X(t),t)\,dt + f'_t(X(t),t)\,dt.$$

In our case $f(x,t) = \Phi(x/\sqrt{T-t}\,)$ and $X(t) = B(t)$, so that $f'_x(x,t) = \phi(x/\sqrt{T-t}\,)/\sqrt{T-t}$, $f''_{xx}(x,t) = \phi'(x/\sqrt{T-t}\,)/(T-t) = -x\,\phi(x/\sqrt{T-t}\,)/(T-t)^{3/2}$ and $f'_t(x,t) = \frac{1}{2}\,x\,\phi(x/\sqrt{T-t}\,)/(T-t)^{3/2}$ giving $d(\Phi(B(t)/\sqrt{T-t}\,)) = \phi(B(t)/\sqrt{T-t}\,)/\sqrt{T-t}\;dB(t)$ (as $\mu(x,t)=0$ and $\sigma(x,t)=1$). And so $\Phi(B(t)/\sqrt{T-t}\,) = \int_0^t \phi(B(s)/\sqrt{T-s}\,)/\sqrt{T-s}\;dB(s)$ is a martingale (as $|\phi(x)|\leq 1$ is a bounded function).

**Task 3.** By Theorem 6.8 in Klebaner we have $f(x,t) = \mathbf{E}\{X(T)^2|X(t)=x\}$ where $X(t)$ solves the Langevin SDE $dX(t) = -X(t)\,dt + dB(t)$. As by Example 5.6 in Klebaner $X(t) = \mathrm{e}^{-t}\big(X(0) + \int_0^t \mathrm{e}^s\,dB(s)\big)$ giving $X(T) = \mathrm{e}^{-T}\big(X(0) + \int_0^T \mathrm{e}^s\,dB(s)\big) = \mathrm{e}^{-(T-t)}X(t) + \mathrm{e}^{-T}\int_t^T \mathrm{e}^s\,dB(s)$ we get $f(x,t) = \mathbf{E}\big\{(\mathrm{e}^{-(T-t)}x + \mathrm{e}^{-T}\int_t^T \mathrm{e}^s\,dB(s))^2\big\} = \mathrm{e}^{-2(T-t)}x^2 + \frac{1}{2}\,(1-\mathrm{e}^{-2(T-t)})$ (where we also made use of Theorem 4.11 in Klebaner).

**Task 4.** We have

$$\begin{aligned} d(X(t)^2) &= 2\,X(t)\,dX(t) + d[X,X](t) \\ &= 2\,X(t)\,\mu(X(t))\,dt + 2\,X(t)\,\sigma(X(t))\,dB(t) + \sigma(X(t))^2\,dt \end{aligned}$$

making

$$X(t)^2 - 2\int_0^t X(s)\,\mu(X(s))\,ds - \int_0^t \sigma(X(s))^2\,ds = 2\int_0^t X(s)\,\sigma(X(s))\,dB(s)$$

a martingale, so we can take $\mu(x) = x/2$ and $\sigma(x)=1$.

**Task 5.** It is clear by inspection that $\mathbf{P} = (\mathbf{P}_1+\mathbf{P}_2)/2$ obeys the axioms of a probability measure on page 29 in Klebaner when $\mathbf{P}_1$ and $\mathbf{P}_2$ do. Further, according to Theorem 10.4

in Klebaner $d\mathbf{P}_2/d\mathbf{P}_1 = \Lambda(X) = (\sigma_1/\sigma_2)\,\mathrm{e}^{\frac{1}{2}(X-\mu_1)^2/\sigma_1^2-\frac{1}{2}(X-\mu_2)^2/\sigma_2^2}$ so that $d\mathbf{P}/d\mathbf{P}_1 = (1+\Lambda(X))/2$, $d\mathbf{P}/d\mathbf{P}_2 = (\Lambda(X)^{-1}+1)/2$, $d\mathbf{P}_1/d\mathbf{P} = 2/(1+\Lambda(X))$ and $d\mathbf{P}_2/d\mathbf{P} = 2/(\Lambda(X)^{-1}+1)$ showing that $\mathbf{P}$ is equivalent to both $\mathbf{P}_1$ and $\mathbf{P}_2$. Also, clearly, $\mathbf{P}\{X \leq 0\} = (\Phi(-\mu_1/\sigma_1)+\Phi(-\mu_2/\sigma_2))/2$.

**Task 6.** This is Section 2.3 in the lecture notes by Stig Larsson.