TMS165/MSA350 Stochastic Calculus

Written Exam Wednesday 4 January 2017 8.30–12.30 am

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Aids: Two sheets (=four pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed).

Grades: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

Motivations: All answers/solutions must be motivated. Good Luck!

Througout this exam $B = \{B(t)\}_{t \geq 0}$ denotes a Brownian motion.

Task 1. Calculate $dX(t)$ for

$$X(t) = (1-t) \int_0^t \frac{dB(s)}{1-s} \quad \text{for } t \in [0,1). \qquad \textbf{(5 points)}$$

Task 2. Consider a simple adapted process $\{X(t)\}_{t \in [0,T]}$ given by

$$X(t) = I_{\{0\}}(t)\xi_0 + \sum_{i=0}^{n-1} I_{(t_i,t_{i+1}]}(t)\xi_i \quad \text{for } t \in [0,T],$$

where $0 = t_0 < t_1 < \ldots < t_n = T$ is a partition of the interval $[0,T]$ and ξ_i is an $\mathcal{F}_{t_i}^B$-measurable random variable with $\mathbf{E}\{\xi_i^2\} < \infty$ for $i = 0, \ldots, n-1$. Prove the isometry property $\mathbf{E}\big\{(\int_0^T X(t)\, dB(t))^2\big\} = \int_0^T \mathbf{E}\{X(t)^2\}\, dt$. **(5 points)**

Task 3. Solve the SDE

$$dX(t) = B(t)\,X(t)\,dt + B(t)\,X(t)\,dB(t) \quad \text{for } t \geq 0, \quad X(0) = 1. \qquad \textbf{(5 points)}$$

Task 4. Define a new kind of stochastic integral of one Itô process $\{Y(t)\}_{t \in [0,T]}$ with respect to another Itô process $\{X(t)\}_{t \in [0,T]}$ as $\int_0^T Y(t)\, \Delta X(t) = \lim \sum_{i=1}^n Y(t_i)\,(X(t_i) - X(t_{i-1})$ for partitions $0 = t_0 < t_1 < \ldots < t_n = T$ of the interval $[0,T]$ that become finer and finer so that $\max_{1 \leq i \leq n} t_i - t_{i-1} \downarrow 0$ in the limit. What is the relation between this new integral and the Itô integral $\int_0^T Y(t)\, dX(t)$? (Remember that answers must be motivated - a correct answer without a motivation gives no points!) **(5 points)**

Task 5. Let $\mathfrak{G}$ be a sub σ-field to the σ-field $\mathfrak{F}$ on which two probability measures $\mathbf{P}$ and $\mathbf{Q}$ are defined. If $\mathbf{Q}$ is absolutely continuous with respect to $\mathbf{P}$ ($\mathbf{Q} << \mathbf{P}$) with $d\mathbf{Q}/d\mathbf{P} = \Lambda$ and X is a random variable with $\mathbf{E}_{\mathbf{Q}}\{|X|\} < \infty$, show that

$$\mathbf{E}_{\mathbf{P}}\{|X|\Lambda\} < \infty \qquad \text{and} \qquad \mathbf{E}_{\mathbf{Q}}\{X\,|\,\mathfrak{G}\} = \frac{\mathbf{E}_{\mathbf{P}}\{X\Lambda\,|\,\mathfrak{G}\}}{\mathbf{E}_{\mathbf{P}}\{\Lambda\,|\,\mathfrak{G}\}}. \qquad \textbf{(5 points)}$$

Task 6. The Euler method for numerical solution of a time-hohogeneous SDE on a grid $0 = t_0 < t_1 < \ldots < t_n = T$ is given by (with obvious notation) $Y_0 = x_0$ and

$$Y_{k+1} = Y_k + \mu(Y_k)\,\Delta t_k + \sigma(Y_k)\,\Delta B(t_k) \quad \text{for } \ k = \{0, \ldots, n-1\}.$$

The Milstein method, on the other hand, is given by

$$Y_{k+1} = Y_k + \mu(Y_k)\,\Delta t_k + \sigma(Y_k)\,\Delta B(t_k) + \tfrac{1}{2}\,\sigma(Y_k)\sigma'(Y_k)\big((\Delta B(t_k))^2 - \Delta t_k\big).$$

Explain why the Milstein method usaually is better (gives a better numerical approxiamtion, that is) than the Euler method. **(5 points)**

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Solutions to Written Exam 4 January 2017

Task 1. We have $X(t) = f(t, Y(t))$ with $f(t, y) = (1-t)\, y$ and $Y(t) = \int_0^t (1-s)^{-1}\, dB(s)$. It follows that

$$dX(t) = f_t'(t, Y(t))\, dt + f_y'(t, Y(t))\, dY(t) + \tfrac{1}{2}\, f_{yy}''(t, Y(t))\, d[Y, Y](t) = -Y(t)\, dt + dB(t).$$

Task 2. See page 94-95 in Klebaner.

Task 3. Clearly, $X(t)$ is the stochastic exponential of the process $Y(t)$ given by $dY(t) = B(t)\, dt + B(t)\, dB(t)$. As $Y(t) = \int_0^t B(s)\, ds + \int_0^t B(s)\, dB(s) = \int_0^t B(s)\, ds + \frac{1}{2}\, B(t)^2 - \frac{1}{2}\, t$ we get $X(t) = \mathrm{e}^{\int_0^t B(s)\, ds + \frac{1}{2} B(t)^2 - \frac{1}{2} t - \frac{1}{2} \int_0^t B(s)^2\, ds}$.

Task 4. We have $\int_0^T Y(t)\, \Delta X(t) - \int_0^T Y(t)\, dX(t) = \lim \sum_{i=1}^n (Y(t_i) - Y(t_{i-1})\, (X(t_i) - X(t_{i-1}) = [Y(t), X(t)]$.

Task 5. This is Theorem 10.8 in Klebaner: See his proof.

Task 6. See Section 2.4 in Stig Larsson's lecture notes.