TMS165/MSA350 Stochastic Calculus

Written Exam Friday 18 August 2017 8.30–12.30 am

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AIDS: Two sheets (=four pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed).

GRADES: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Througout this exam $B = \{B(t)\}_{t\geq 0}$ denotes a Brownian motion.

Task 1. Find the quadratic variation process of $\cos(B(t))\,\mathrm{e}^{B(t)}$. **(5 points)**

Task 2. If $X(t)$ is the stochastic exponential of $B(t)$, what process is then $X(t)^3$ the stochastic exponential of? **(5 points)**

Task 3. We have observed a diffusion process $\{X(t)\}_{t\in[0,10]}$ which is either Brownian motion $X(t) = B(t)$ or an Ornstein-Uhlenbeck process $dX(t) = -X(t)\,dt + dB(t)$. How can we use the observed data $\{X(t)\}_{t\in[0,10]}$ to determine which of the two models (/origins) for $X(t)$ is the correct one? **(5 points)**

Task 4. For which two times continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ is $\{f(B(t)+t)\}_{t\geq 0}$ a time-homogeneous diffusion process with zero drift coefficient? **(5 points)**

Task 5. Let $\{X(t)\}_{t\geq 0}$ be a time homogeneous diffusion process that has a stationary distribution and is started according to that stationary distribution at time $t = 0$. Prove that X is a stationary process, which is to say that

$$\mathbf{P}\big\{X(t_1+h) \leq x_1, \ldots, X(t_n+h) \leq x_n\big\} = \mathbf{P}\big\{X(t_1) \leq x_1, \ldots, X(t_n) \leq x_n\big\}$$

for $0 < t_1 < \ldots < t_n$, $h \geq 0$ and $x_1, \ldots, x_n \in \mathbb{R}$. **(5 points)**

Task 6. Explain the difference between strong convergence and weak convergence of a numerical method for solution of SDE's. **(5 points)**

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Solutions to Written Exam 18 August 2017

Task 1. As $d(\cos(B(t))\,\mathrm{e}^{B(t)}) = -\sin(B(t))\,\mathrm{e}^{B(t)}\,dB(t) + \cos(B(t))\,\mathrm{e}^{B(t)}\,dB(t) + \ldots\, dt$ we have $[\cos(B(t))\,\mathrm{e}^{B(t)}, \cos(B(t))\,\mathrm{e}^{B(t)}] = \int_0^t (\sin(B(s) - \cos(B(s)))^2 \mathrm{e}^{2B(s)}\,ds$.

Task 2. As $dX(t) = X(t)\,dB(t)$ we have that $Y(t) = X(t)^3$ satisfies $dY(t) = d(X(t)^3) = 3\,X(t)^2\,dX(t) + 3\,X(t)\,d[X,X](t) = 3\,X(t)^3\,dB(t) + 3\,X(t)^3\,dt = Y(t)\,(3\,dB(t) + 3\,dt)$, making $X(t)^3$ the stochastic exponential of $3\,B(t) + 3t$.

Task 3. According to Example 10.5 and Equation 10.53 in Klebaner's book we calculate the likelihood

$$\Lambda(X) = \exp\bigg\{ -\int_0^{10} X(t)\,dX(t) - \frac{1}{2}\int_0^{10} X(t)^2\,dt \bigg\}.$$

If this likelihood is (significantly) bigger that 1 we conclude that $X(t)$ is an Ornrstein-Uhlenbeck process while if the likelihood is (significantly) smaller that 1 we conclude that $X(t)$ is a Brownian motion.

Task 4. As $X(t) = f(B(t){+}t)$ satisfies $dX(t) = f'(B(t){+}t)\,(dB(t){+}dt) + \frac{1}{2}\,f''(B(t){+}t)\,dt$ we must have $f''(x) + 2f'(x) = 0$ so that $f'(x) + 2f(x) = C_1$ giving $\frac{d}{dx}\,(\mathrm{e}^{2x} f(x)) = C_1\,\mathrm{e}^{2x}$ so that $\mathrm{e}^{2x} f(x) = \frac{1}{2}\,C_1\,\mathrm{e}^{2x} + C_2$ and $f(x) = C_1 + C_2\,\mathrm{e}^{-2x}$ (for another choise of C_1).

Task 5. Writing $\pi(x)$ for the stationary probability density function and supposing that $X(0)$ has that density function, we have with obvious notation using Eq. 6.67 in Klebaner's book (see also Eq. 3.4 in Klebaner's book)

$$\begin{aligned} &\mathbf{P}\big\{X(t_1{+}h) \le x_1, \ldots, X(t_n{+}h) \le x_n\big\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{x_1}\ldots\int_{-\infty}^{x_n} \pi(x)\,p(t_1{+}h, x, y_1)\,p(t_2{-}t_1, y_1, y_2)\,\ldots\,p(t_n{-}t_{n-1}, y_{n-1}, y_n)\,dy_n \ldots dy_1 dx \\ &= \int_{-\infty}^{x_1}\ldots\int_{-\infty}^{x_n} \pi(y_1)\,p(t_2{-}t_1, y_1, y_2)\,\ldots\,p(t_n{-}t_{n-1}, y_{n-1}, y_n)\,dy_n \ldots dy_1. \end{aligned}$$

As the right-hand side of this expression does not depend on $h \ge 0$ we are done.

Task 6. See the beginning of Chapter 2 in the lecture notes by Stig Larsson.