TMS165/MSA350 Stochastic Calculus

Written Exam Friday 24 August 2018 8.30–12.30

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AIDS: Two sheets (=four pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed).

GRADES: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Througout this exam $B = \{B(t)\}_{t\geq 0}$ denotes a Brownian motion.

Task 1. Let X be an Itô process with stochastic exponential $\mathcal{E}(X)$ and quadratic variation $[X,X]$. Find an expression for the covariation $[X,\mathcal{E}(X)]$. **(5 points)**

Task 2. A fundamental feature of stochastic calculus as opposed to ordinary calculus is that infinitesimal calculus has to be carried out to the second order instead of just the first. Explain why this is so. **(5 points)**

Task 3. There is a well-known method/formula for solving the following SDE:

$$dX(t) = dt + X(t)\,dB(t) \quad \text{for } t \geq 0, \quad X(0) = 0.$$

What method/formula is that? [You don't have to solve the SDE but just name the method/formula. However, should you not be familiar with (the name) of the method/ formula you can get full score on the task by solving the SDE yourself.] **(5 points)**

Task 4. Given a constant $\alpha \neq 0$, determine whether the solution to the following SDE has a stationary probability density function:

$$dX(t) = X(t)\,dt + \frac{1}{\alpha}\sqrt{1+X(t)^2}\,dB(t) \quad \text{for } t \geq 0, \quad X(0) = 0.$$ **(5 points)**

Task 5. Let X be a continuous random variable defined on a probability space $(\Omega,\mathcal{F},\mathbf{P})$ with probability density function $f(x) > 0$ for all $x \in \mathbb{R}$. Explain how it is possible to give X any other (continuous random variable) probability density function $g(x)$ by means of switching to another probability measure $\mathbf{Q}$ on $(\Omega,\mathcal{F})$ (=the sample space together with its measurable sets/events) than the original probability measure $\mathbf{P}$.

(5 points)

Task 6. The Euler method is the simplest possible scheme that exists for approximate numerical strong solution of SDE. But there are also exist more complicated higher order method such as, e.g., the Milstein method. Explain what is the difference between the Euler method and higher order methods and also indicate what are their respective advantages and disadvantages. **(5 points)**

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Solutions to Written Exam 24 August 2018

Task 1. $d[X,\mathcal{E}(X)] = dX\, d\mathcal{E}(X) = dX\,\mathcal{E}(X)\,dX = \mathcal{E}(X)\,d[X,X]$ so that $[X,\mathcal{E}(X)](t) = \int_0^t (\mathcal{E}(X))(s)\,d[X,X](s)$.

Task 2. It is because stochastic calculus is infenitesimal calculus with Itô processes $dX(t) = \mu(t)\,dt + \sigma(t)\,dB(t)$ where $(dX(t))^2 = \sigma(t)^2\,dt$ is of the same order as $dX(t)$ and can thus not be neglected.

Task 3. The formula for solution of a general linear SDE can be applied (which in turn is a combination of the method of stochastic exponential with that of integration by parts).

Task 4. If the stationary probability density function exists it is given by

$$\frac{C}{\sigma(x)^2}\exp\left\{\int_0^x \frac{2\mu(y)}{\sigma(y)^2}\,dy\right\} = \frac{C\alpha^2}{1+x^2}\exp\left\{\int_0^x \frac{2\alpha^2 y}{1+y^2}\,dy\right\} = \frac{C\alpha^2}{1+x^2}\,\mathrm{e}^{\alpha^2\ln(1+x^2)} = \frac{C\alpha^2}{(1+x^2)^{1-\alpha^2}}$$

for some constant $C > 0$. Hence it follows that the stationary probability density function exists if and only if $\alpha \in (0, 1/\sqrt{2}\,)$. (And in that case we have an example of so called VIS - volatility induced stationarity.)

Task 5. Defining $Q(A) = \int_\Omega (g(X)/f(X))\,I_A\,d\mathbf{P} = \mathbf{E_P}\{(g(X)/f(X))\,I_A\}$ for $A \in \mathcal{F}$ it is easy to see that $\mathbf{Q}$ is a probability measure with $\mathbf{E_Q}\{Y\} = \int_\Omega (g(X)/f(X))\,Y\,d\mathbf{P}$ for all random variables Y so that $\mathbf{Q}(\{X\in A\}) = \mathbf{Q}(\{\omega\in\Omega : X(\omega)\in A\}) = \mathbf{E_Q}\{I_A(X)\} = \int_\omega (g(X)/f(X))\,I_A(X)\,d\mathbf{P} = \int_{x\in\mathbb{R}} (g(x)/f(x))\,I_A(x)\,f(x)\,dx = \int_A g(x)\,dx$ giving X probability density function $g(x)$ under $\mathbf{Q}$.

Task 6. This is explained in Stig's lecture notes.