TMS165/MSA350 Stochastic Calculus

Written Exam Tuesday 30 October 2018 8.30–12.30

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AIDS: Two sheets (=four pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed).

GRADES: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Througout this exam $B = \{B(t)\}_{t\geq 0}$ denotes a Brownian motion.

Task 1. Let X be an Itô process with stochastic exponential $\mathcal{E}(X)$ and quadratic variation $[X,X]$. Show that $\mathcal{E}(2X) = (\mathcal{E}(X))^2$ if and only if X has finite variation.

(5 points)

Task 2. State and prove the integration by parts formula for two Itô processes X and Y. **(5 points)**

Task 3. Do $X(t) = \sin(B(t))$ satisfy any SDE? **(5 points)**

Task 4. **(a)** Give an example of an SDE that explodes. **(1.25 points)**

(b) Give an example of an SDE that is transient but does not explode. **(1.25 points)**

(c) Give an example of an SDE that is reccurent but does not have a stationary distribution. **(1.25 points)**

(d) Give an example of an SDE that has a stationary distribution. **(1.25 points)**

Note: The answers to tasks (a)-(d) must be motivated – just an answer is not enough.

Task 5. A student collegue of yours has used the Euler method to create an approximate numerical solution $\{Y(t)\}_{t\in[0,1]}$ to the SDE

$$dX(t) = (\alpha + \beta X(t))\, dt + \sigma\, dB(t) \quad \text{for } t \in [0,1], \quad X(0) = 0,$$

where $\alpha, \beta \in \mathbb{R}$ and $\sigma > 0$ are parameters the values of which are known only to your collegue. How can you estimate these parameters from the approximate numerical solution $\{Y(t)\}_{t\in[0,1]}$? **(5 points)**

Task 6. **(a)** What are the convergence properties of the Euler method for approximate strong numerical solution of SDE? What are the convergence properties of the Euler method for approximate weak numerical solution of SDE? (These two questions just have to be answered – no motivation is required.) **(3 points)**

(b) The strong and weak convergence properties in task (a) are different, as you should be well aware of. Say something about why they can be different despite that it is the one and same method that is used. **(2 points)**

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Solutions to Written Exam 30 October 2018

Task 1. We have $\mathcal{E}(2X)(t)/(\mathcal{E}(X)(t))^2 = \mathrm{e}^{\frac{3}{2}[X,X]](t)} = 1$ if and only if $[X,X](t) = 0$ which happens if and only if the Itô integral part of the Itô process is constant which means that the Itô process is finite variation.

Task 2. See page 113 in Klebaner's book.

Task 3. We have $d(\sin(B(t)) = \cos(B(t))\,dB(t) - \frac{1}{2}\sin(B(t))\,dt = \cos(B(t))\,dB(t) - \frac{1}{2}X(t)\,dt$ with $X(0) = 0$ (which is a non diffusion type SDE). Here it is tempting to write $\cos(B(t)) = \sqrt{1+\sin(B(t))^2} = \sqrt{1+X(t)^2}$ to arrive at the diffusion type SDE $dX(t) = -\frac{1}{2}X(t)\,dt + \sqrt{1+X(t)^2}\,dB(t)$ with $X(0)=0$. However, as we can really only know that $\cos(B(t)) = \pm\sqrt{1+X(t)^2}$ this is not correct. Still we give almost full score for the answer $dX(t) = -\frac{1}{2}X(t)\,dt + \sqrt{1+X(t)^2}\,dB(t)$ with $X(0)=0$ as well.

Task 4. **(a)** $dX(t) = X(t)^2\,dt$ with $X(0) = 1$ has solution $X(t) = 1/(1-t)$.

(b) $dX(t) = X(t)\,dt$ with $X(0) = 1$ has solution $X(t) = \mathrm{e}^t$.

(c) BM is well-known to be reccurent but having no stationary distribution.

(d) The Ornstein-Uhlenbeck process is well-known to have a stationary distribution.

Task 5. As $[X,X](1) = \sigma^2$ one can estimate σ^2 with $\sum_{i=1}^n (Y(t_i)-Y(t_{i-1}))^2$, where $0=t_0<t_1<\ldots<t_n=1$ with $\max_{1\le i\le n} t_i-t_{i-1}$ very small. Further the likelihood ratio technique from Section 10.6 in Klebaner's book ensures that α and β can be estimated by the particular values of these parameters that minimize the likelihood ratio

$$\exp\bigg\{\frac{1}{\sigma^2}\int_0^1(\alpha+\beta Y(t))\,dY(t) - \frac{1}{2\sigma^2}\int_0^1(\alpha+\beta Y(t))^2\,dt\bigg\}.$$

Task 6. **(a)** See Stig Larssons's lecture notes.

(b) The reason is that strong convergence is measured by means of a much more demanding requirement than weak convergence.