TMS165/MSA350 Stochastic Calculus

Written Exam Friday 4 January 2019 2–6 pm

TEACHER: Patrik Albin. JOUR: Jimmy Johansson, telephone 031 7725325.

AIDS: Two sheets (=four pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed).

GRADES: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Througout this exam $B = \{B(t)\}_{t\geq 0}$ denotes a Brownian motion.

Task 1. Find the quadratic variation process of $\{B(f(t))\}_{t\geq 0}$ when $f : [0,\infty) \to [0,\infty)$ is a continuously differentiable strictly increasing function. **(5 points)**

Task 2. Do $X(t) = \sinh(B(t)+t)$ satisfy any SDE? [Recall that $\sinh(t) = \frac{1}{2}(\mathrm{e}^t - \mathrm{e}^{-t})$ with $\frac{d}{dt}\sinh(t) = \cosh(t) = \frac{1}{2}(\mathrm{e}^t + \mathrm{e}^{-t})$ and $\cosh(t)^2 = 1 + \sinh(t)^2$.] **(5 points)**

Task 3. Let X be a Itô process with stochastic exponential $\mathcal{E}(X)$ and stochastic logarithm $\mathcal{L}(X)$ (for X strictly positive). Is it true that $\mathcal{E}(\mathcal{L}(X)) = X$ or that $\mathcal{L}(\mathcal{E}(X)) = X$? **(5 points)**

Task 4. Given constants $\alpha, \sigma > 0$, find the solution $f(x,t)$ to the PDE

$$\frac{\partial f(x,t)}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 f(x,t)}{\partial x^2} - \alpha x \frac{\partial f(x,t)}{\partial x} = 0 \quad \text{for } t \in [0,T], \quad f(x,T) = x. \qquad \textbf{(5 points)}$$

Task 5. Let $H \in P_T$ be such that the stochastic exponential $\mathcal{E}(X)$ of $X(t) = -\int_0^t H\,dB$ is a martingale. Show that $\{B(t) + \int_0^t H(s)\,ds\}_{t\in[0,T]}$ is a Brownian motion under the probability measure $\mathbf{Q}$ given by $d\mathbf{Q}/d\mathbf{P} = \mathcal{E}(X)(T)$. **(5 points)**

Task 6. Describe how Itô-Taylor expansion is made and say something about why it is made. **(5 points)**

TMS165/MSA350 Stochastic Calculus

Solutions to Written Exam January 2019

Task 1. $\{f(t)\}_{t\geq 0}$.

Task 2. For $X(t) = \sinh(B(t)+t)$ we have $dX(t) = \cosh(B(t)+t)\,(dB(t)+dt) + \frac{1}{2}\sinh(B(t)+t)\,dt = \big(\sqrt{1+X(t)^2} + \frac{1}{2}X(t)\big)\,dt + \sqrt{1+X(t)^2}\,dB(t)$ with $X(0)=0$.

Task 3. As for an Itô process Y we have $\mathcal{E}(Y)(0) = 1$ and $\mathcal{L}(Y)(0) = 0$ (for Y strictly positive) it cannot be true in general that $\mathcal{E}(\mathcal{L}(X))(t) = X(t)$ or that $\mathcal{L}(\mathcal{E}(X))(t) = X(t)$ [as the first identity requires $X(0) = 1$ and the second that $X(0) = 0$].

Task 4. According to the Feynman-Kac formula we have $f(x,t) = \mathbf{E}\{X(T)\,|\,X(t) = x\}$ where $X(t) = \mathrm{e}^{-\alpha t}\big(X(0) + \int_0^t \sigma\,\mathrm{e}^{\alpha r}\,dB(r)\big) = \mathrm{e}^{-\alpha(t-s)}X(s) + \mathrm{e}^{-\alpha t}\int_s^t \sigma\,\mathrm{e}^{\alpha r}\,dB(r)$ solves the Langevin equation so that $f(x,t) = \mathbf{E}\{\mathrm{e}^{-\alpha(T-t)}x + \mathrm{e}^{-\alpha T}\int_t^T \sigma\,\mathrm{e}^{\alpha r}\,dB(r)\} = \mathrm{e}^{-\alpha(T-t)}x$.

Task 5. See the proof of Theorem 10.16 in Klebaner's book.

Task 6. See Stig Larsson's lecture notes.