TMS 165/MSA350 Stochastic Calculus,

Selections from and additions to Klebaner's book

Chapter 1

Section 1.2

Equations 1.7-1.9 [we use (1.9) instead of (1.7) although (1.9) is more general].

The notation $V_g(t)$ and the fact that this is a non-decreasing function.

Example 1.4.

Example 1.5.

Example 1.6.

Theorem 1.6.

Equation 1.13.

Theorem 1.10.

Equation 1.15.

Theorem 1.11.

Theorem 1.12.

Equation 1.17.

Section 1.3

Equations 1.18 and 1.19.

The "Particular Cases" on top of page 11.

The integration by parts formula on last row of page 12.

Chapter 2

Section 2.2

The definition of σ-field on the bottom of page 28.

The definition of probability on the middle of page 29.

The definition of random variable on the middle of page 30.

Example 2.8.

The definition of σ-field generated by random variable on the lower part of page 31.

Section 2.3

The first formula for $\mathbf{E}\{X\}$ of Section 2.3.

The definition of the Lebesgue integral on the lower part of page 33 (including Example 2.9 on page 31).

Equation 2.6.

The Properties 1-3 of expectation on page 35.

Section 2.4

Theorems 2.16-2.18.

Sections 2.1 and 2.2

The definition of a σ-field of events (revisited).

Example 2.1.

The definition of a filtration.

Definition 2.1.

The definition of the σ-field generated by a random variable (revisited).

The definition of the filtration generated by a stochastic process.

Section 2.7

Everything from "General Conditional Expectation" on page 44 up to and including Theorem 2.24 on page 46.

Section 2.8

Definition 2.30.

Chapter 3

Section 3.1

The definition of Brownian motion (BM) on page 56 - note the unspecificness of the value for $B(0)$.

Example 3.2.

Equation 3.3.

Equation 3.4 without proof.

The notation B^x and Equation 3.5.

Definition 3.1 applied to BM.

Figure 3.1.

The definition of a Gaussian process on page 59.

The fact that BM is a Gaussian process.

Definition 3.2.

Theorem 3.3.

Example 3.4.

Section 3.2

Everything from the subsection “Quadratic variation of BM”.

Everything from the subsection “Properties of Brownian paths”.

Section 3.3

Theorem 3.7.

Section 3.4

Definition 3.8.

Theorem 3.9.

The definition of the transition probability at the bottom of page 67.

(Stopping times we will introduce later when they are needed.)

Sections 3.5-3.14

This material is not included in the course.

Chapter 4

Definition 2.11.

Definition 2.12.

Theorem. (Cauchy criterion) *A sequence* $\{X_n\}_{n=1}^{\infty}$ *of random variables converges in probability to some random variable* X *if and only if*

$$\lim_{m,n\to\infty} \mathbf{P}\big\{|X_n - X_m| > \varepsilon\big\} = 0 \quad \textit{for each } \varepsilon > 0.$$

Definition 2.13.

Definition 2.14.

Theorem. (Cauchy criterion) *A sequence* $\{X_n\}_{n=1}^{\infty}$ *of random variables such that* $\mathbf{E}\{|X_n|^r\} < \infty$ *for all* n *converges in* $\mathbb{L}^r$ *to some random variable* X *if and only if*

$$\lim_{m,n\to\infty} \mathbf{E}\big\{|X_n - X_m|^r\big\} = 0.$$

Sections 4.1 - 4.2

Definition 4.2 - we use S_T to denote the class of simple adapted processes $\{X(t)\}_{t\in[0,T]}$.

The Itô integral process $\{\int_0^t X\,dB\}_{t\in[0,T]}$ for $X \in S_T$ given by Equation 4.4.

Properties 1-4 of the Itô integral process for S_T on the bottom of page 93 and the top pf page 94. Further, we mention the martingale property (Theorem 4.7) as well as continuity and adaptedness of the integral process for S_T already here.

Definition. *The class* E_T *consists of all adapted processes* $\{X(t)\}_{t\in[0,T]}$ *that satisfies*

$$\mathbf{E}\bigg\{\int_0^T X(t)^2\,dt\bigg\} < \infty.$$

Theorem. *For $X \in E_T$ there exists a sequence $\{X_n\}_{n=1}^{\infty} \subseteq S_T$ such that*

$$\lim_{n\to\infty} \mathbf{E}\left\{\int_0^T (X_n(t) - X(t))^2 \, dt\right\} = 0.$$

Theorem and Definition. *For $X \in E_T$ the Itô integral process $\{\int_0^t X \, dB\}_{t\in[0,T]}$ is well-defined and defined as a limit in the sense of convergence in $\mathbb{L}^2$ of $\int_0^t X_n \, dB$ as $n \to \infty$ for each $t \in [0,T]$, where $\{X_n\}_{n=1}^{\infty} \subseteq S_T$ are as in the previous theorem*

Definition. *The class E_T consists of all adapted processes $\{X(t)\}_{t\in[0,T]}$ that satisfies*

$$\mathbf{E}\left\{\int_0^T X(t)^2 \, dt\right\} < \infty.$$

Theorem. *For $X \in E_T$ there exists a sequence $\{X_n\}_{n=1}^{\infty} \subseteq S_T$ such that*

$$\lim_{n\to\infty} \mathbf{E}\left\{\int_0^T (X_n(t) - X(t))^2 \, dt\right\} = 0.$$

Proof when X is continuous[1]. Given $X \in E_T$ and $\varepsilon > 0$ we need to prove that

$$\mathbf{E}\left\{\int_0^T (Y(t) - X(t))^2 \, dt\right\} \le \varepsilon \quad \text{for some } Y \in S_T.$$

To that end let

$$X^{(N)}(t) = \begin{cases} -N & \text{if} \quad X(t) < -N \\ X(t) & \text{if} \quad |X(t)| \le N \\ N & \text{if} \quad X(t) > N \end{cases}.$$

Since $X^{(N)}(t) - X(t) \to 0$ as $N \to \infty$ with $(X^{(N)}(t) - X(t))^2 \le X(t)^2$ we then have

$$\mathbf{E}\left\{\int_0^T \left(X^{(N)}(t) - X(t)\right)^2 dt\right\} \to 0 \quad \text{as } N \to \infty$$

(by dominated convergence Theorem 2.18 in Klebaner's book as $X \in E_T$). Using the elementary inequality $(x+y)^2 \le 2\,x^2 + 2\,y^2$ it follows that it is enough to prove that given $X \in E_T$, $\varepsilon > 0$ and $N \in \mathbb{N}$ we have

$$\mathbf{E}\left\{\int_0^T \left(Y(t) - X^{(N)}(t)\right)^2 dt\right\} \le \varepsilon \quad \text{for some } Y \in S_T.$$

But as $X^{(N)}$ is uniformly continuous over $[0,T]$ (since X is uniformly continuous over $[0,T]$) we have that $Z^{(n)} \in S_T$ given by

$$Z^{(n)}(t) = I_{\{0\}}(t)\, X^{(N)}(0) + \sum_{i=0}^{n-1} I_{(t_i,t_{i+1}]}(t)\, X^{(N)}(t_i) \quad \text{for } t \in [0,T]$$

[1] The proof for a general not necessarily continuous X is exceptionally difficult and is not really required by us as we will later restrict ourselves to continuous X'es only

(where $0=t_0<t_1<\ldots<t_n=T$ as usual) satisfies

$$\sup_{t\in[0,T]}\left|Z^{(n)}(t)-X^{(N)}(t)\right| \leq \sup_{s,t\in[0,T],\,|s-t|\leq \max_{1\leq i\leq n} t_i-t_{i-1}} \left|X^{(N)}(s)-X^{(N)}(t)\right| \to 0$$

as $\max_{1\leq i\leq n} t_i-t_{i-1}\downarrow 0$. In particular $Z^{(n)}(t)-X^{(N)}(t)\to 0$ as $\max_{1\leq i\leq n} t_i-t_{i-1}\downarrow 0$ with $(Z^{(n)}(t)-X^{(N)}(t))^2\leq 4\,N^2$, so that

$$\mathbf{E}\bigg\{\int_0^T \big(Z^{(n)}(t)-X^{(N)}(t)\big)^2\,dt\bigg\}\to 0 \quad \text{as} \quad \max_{1\leq i\leq n} t_i-t_{i-1}\downarrow 0$$

(by dominated convergence Theorem 2.18 in Klebaner's book). Hence we may pick $Y=Z^{(n)}$ for n large enough to make $\max_{1\leq i\leq n} t_i-t_{i-1}$ small enough. $\square$

Theorem and Definition. *For $X\in E_T$ the Itô integral process $\{\int_0^t X\,dB\}_{t\in[0,T]}$ is well-defined and defined as a limit in the sense of convergence in $\mathbb{L}^2$ of $\int_0^t X_n\,dB$ as $n\to\infty$ for each $t\in[0,T]$, where $\{X_n\}_{n=1}^\infty\subseteq S_T$ are as in the previous theorem.*

Proof. We have to show that $\{\int_0^t X_n\,dB\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{L}^2$. But this follows from the isometry property for the Itô integral for S_T as

$$\begin{aligned}
&\mathbf{E}\bigg\{\bigg(\int_0^t X_n\,dB-\int_0^t X_m\,dB\bigg)^2\bigg\}\\
&\qquad = \mathbf{E}\bigg\{\bigg(\int_0^t (X_n-X_m)\,dB\bigg)^2\bigg\}\\
&\qquad = \mathbf{E}\bigg\{\int_0^t (X_n(t)-X_m(t))^2\,dt\bigg\}\\
&\qquad \leq 2\,\mathbf{E}\bigg\{\int_0^t (X_n(t)-X(t))^2\,dt\bigg\}+2\,\mathbf{E}\bigg\{\int_0^t (X(t)-X_m(t))^2\,dt\bigg\}\\
&\qquad \to 0 \quad \text{as } m,n\to\infty.
\end{aligned}$$

We must also show that if also $\{\hat{X}_n\}_{n=1}^\infty\subseteq S_T$ satisfies

$$\lim_{n\to\infty}\mathbf{E}\bigg\{\int_0^T (\hat{X}_n(t)-X(t))^2\,dt\bigg\}=0,$$

so that $\int_0^t \hat{X}_n\,dB$ converges in $\mathbb{L}^2$ to some limit $\oint_0^t X\,dB$ as $n\to\infty$ (by what we have just proven), then $\int_0^t X\,dB=\oint_0^t X\,dB$. However, this follows from noting that

$$\begin{aligned}
&\mathbf{E}\bigg\{\bigg(\int_0^t X\,dB-\oint_0^t X\,dB\bigg)^2\bigg\}\\
&\quad =\lim_{n\to\infty}\mathbf{E}\bigg\{\bigg(\int_0^t X_n\,dB-\int_0^t \hat{X}_n\,dB\bigg)^2\bigg\}\\
&\quad \leq 2\lim_{n\to\infty}\mathbf{E}\bigg\{\int_0^T \big(X_n(t)-X(t)\big)^2\,dt\bigg\}+2\lim_{n\to\infty}\mathbf{E}\bigg\{\int_0^T \big(X(t)-\hat{X}_n(t)\big)^2\,dt\bigg\}\\
&\quad =0.
\end{aligned}$$

$\square$

Properties of the Itô integral process for E_T are exactly the same as those for S_T

Definition. *The class P_T consists of all adapted processes $\{X(t)\}_{t\in[0,T]}$ that satisfies*

$$\mathbf{P}\bigg\{\int_0^T X(t)^2\,dt<\infty\bigg\}=1.$$

Theorem. *For $X\in P_T$ we have in the sense of convergence in probability*

$$\int_0^T (X_n(t)-X(t))^2\,dt\to 0 \quad \text{as } n\to\infty \quad \text{for some sequence } \{X_n\}_{n=1}^\infty\subseteq E_T.$$

Theorem and Definition. *For $X\in P_T$ the Itô integral process $\{\int_0^t X\,dB\}_{t\in[0,T]}$ is well-defined and defined as a limit in the sense of convergence in probability of $\int_0^t X_n\,dB$ as $n\to\infty$ for each $t\in[0,T]$, where $\{X_n\}_{n=1}^\infty\subseteq E_T$ are as in the previous theorem.*

Theorem. *A continuous and adapted process $\{X(t)\}_{t\in[0,T]}$ belongs to P_T and satisfies*

$$\sup_{t\in[0,T]}\bigg|\int_0^t X\,dB-\int_0^t\sum_{i=1}^n X(t_{i-1})I_{(t_{i-1},t_i]}\,dB\bigg|\to 0 \quad \text{in probability}$$

for partitions $0=t_0<t_1<\ldots<t_n=T$ of $[0,T]$ such that $\max_{1\leq i\leq n} t_i-t_{i-1}\downarrow 0$.

Properties of the Itô integral process for P_T are the same as those for E_T, except that the properties zero-mean, martingale and isometry need no longer hold.

Example 4.2.

Example 4.5.

Theorem 4.9.

Section 4.3

Theorem 4.11.

Example 4.10.

Section 4.4

Theorem 4.13.

Example 4.12.

Example 4.13.

Section 4.5

Definition of an Itô process (not same thing as an Itô integral process) and of a stochastic differential.

Example 4.14.

Example 4.15.

Equation 4.42.

Everything in the subsection “Integral with respect to Itô processes” (i.e., Equations 4.48-4.50).

Section 4.6

Theorem 4.16.

Example 4.20.

Example 4.23.

Theorem 4.17.

Theorem 4.18.

Example 4.26.

Chapter 5

Section 5.1

Everything in the subsection “Ordinary Differential Equations” on the bottom of page 123 and the top of page 124.

Everything in the subsection “Stochastic Differential Equations” on page 126 until (but not including) Example 5.3.

Example 5.5 and Example 5.3 as a special case there of.

Example 5.6.

Section 5.2

The definition Equation 5.17 of stochastic exponential.

Theorem 5.2.

The definition of stochastic logarithm just before Theorem 5.3.

Theorem 5.3. (A detailed proof of this theorem is given in Exercise session 4.)

Example 5.10.

Section 5.3

The definition of a general linear SDE in Equation 5.23.
The general linear SDE is solved explicitely in the book, but omit the details of these calculations until possibly needed later.

Derive the solution to the Langevin type from the solution to the general linear SDE.

Section 5.4

Theorem 5.4. (A proof of this result is more or less included in the course material on numerical solutions of SDE.)

Theorem 5.5.

Note that Theorems 5.4 and 5.5 only give sufficient conditions for existence and uniqueness of strong solutions to SDE, but that many a specific SDE may display such existence and uniqueness without these sufficient conditions being satisfied as they are in fact very far from necessary.

Example 5.12.

Section 5.5

The Markov property Equation 5.42.

The transition probability Equation 5.43.

Theorem 5.6, with a motivation from the Euler scheme.

We will se a lot more about the Markov property of SDE in the lectures on applications and on numerical methods for SDE.

Section 5.6

Definition 5.8.

Definition 5.9.

Example 5.15.

Section 5.7

Theorems 5.10 and 5.11 as examples of such theorems.

The genarator L_s defined by Equation 5.51.

Definition 5.12.

Theorem 5.13 with soft proof.

Example 5.17.

Section 5.8

Definition 5.14, Theorem 5.15 and Theorem 5.16 stripped of all their technicalities (more or less).

Section 5.9

Equation 5.66.

Definition 5.17.

Theorem 5.18.

Theorem 5.19. (A detailed proof under weaker conditions is given in Exercise session 4.)

Theorem 5.20.

Chapter 6

Section 6.1

The genarator L_t defined by Equation 6.2.

Theorem 6.2 with soft proof.

Corollary 6.4.

Example 6.2.

Section 6.2

Theorem 6.8.

Section 6.3

Definition of a time homogeneous SDE in Equation 6.25.

Theorem 6.13.

Corollary 6.14.

Example 6.8.

Sections 6.4-6.5

This material is not included in the course.

Section 6.6

Definition 6.22.

Theorem 6.23.

Corollary 6.24.

Section 6.7

Definitions 6.25-6.26.

Theorem 6.27.

Theorem 6.28.

Section 6.8

This material is not included in the course.

Section 6.9

Equations 6.66-6.69.

Example 6.15.

Example 6.16.

Chapter 10

Section 10.1

Theorem 10.4.

Theorems 10.2 and 10.3 as special cases of Theorem 10.4.

Example 10.1.

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Example 10.1 in Klebaner ' s book with n = 10 000 000
N[CDF[NormalDistribution[6, 1], 0]]
9.86588×10^-10
Clear[rep, x]; rep = 10 000 000;
Sum[{x = Random[NormalDistribution[0, 1]], If[x < 0, 1, 0] * Exp[6 * x - 18]}[[2]], {i, 1, rep}] /rep
9.86475×10^-10
```

Section 10.2

Everything except Example 10.3.

Section 10.3

Everything on page 277.

Equations 10.31-10.35 from the subsection "Change of Drift in Diffusions".

Theorems 10.15 and 10.16 as special cases of change of drift in diffusions.

Section 10.6

Everything from the subsection "Likelihood Ratios for Diffusions" (i.e., Equations 10.49-10.60).