TMS165/MSA350 Stochastic Calculus

Written Exam Friday 23 August 2019 8.30 – 12.30

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AIDS: Two sheets (=four pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed).

GRADES: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Througout this exam $B = \{B(t)\}_{t\geq 0}$ denotes a Brownian motion.

Task 1. For two functions $f, g : [0,\infty) \to \mathbb{R}$ we know the values of the quadratic variations $[f+g, f+g](t)$ and $[f-g, f-g](t)$ at a time $t > 0$. Based on this information only (but knowing nothing more than that about f and g), which of the quantities $[f,f](t)$, $[g,g](t)$ and $[f,g](t)$ can we calculate the value of (if any)? **(5 points)**

Task 2. Find the solution $\{X(t)\}_{t\geq 0}$ to the SDE

$$dX(t) = X(t)\,\sin(B(t))\,dB(t) \quad \text{for } t > 0, \quad X(0) = 1.$$

(5 points)

Task 3. Find a solution $p(y,t)$ to the PDE

$$\frac{1}{2}\frac{\partial^2 p}{\partial y^2} - \frac{\partial p}{\partial y} - \frac{\partial p}{\partial t} = 0 \quad \text{for } (y,t) \in \mathbb{R} \times (0,\infty)$$

that is a probability density function as a function of y for any given t. **(5 points)**

Task 4. Let X be a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ [where Ω is the sample space of all possible outcomes of a random experiment, $\mathcal{F}$ is the σ-field of those subsets of Ω that are events and $\mathbf{P}$ is a probability measure defined on $(\Omega, \mathcal{F})$]. Prove that $\mathbf{E}\{\mathbf{E}\{X\,|\,\mathcal{G}\}\} = \mathbf{E}\{X\}$ for any σ-field $\mathcal{G}$ that is contained in $\mathcal{F}$ and that $\mathbf{E}\{X\,|\,\mathcal{G}\} = \mathbf{E}\{X\}$ in the particular case when $\mathcal{G} = \{\emptyset, \Omega\}$. **(5 points)**

Task 5. Given some constants $\mu, \sigma \in \mathbb{R}$, consider the SDE

$$dX(t) = \mu\,dt + \sigma X(t)\,dB(t) \quad \text{for } t \geq 0.$$

Does this SDE have a stationary distribution? If the answer is yes, find that stationary distribution – if the answer is no, explain why not. **(5 points)**

Task 6. Consider a so called fully implicit numerical scheme given by

$$Y_0 = 1 \quad \text{and} \quad Y_k = Y_{k-1} - Y_k\,(t_k - t_{k-1}) + Y_k\,(B(t_k) - B(t_{k-1})) \quad \text{for } k = \{1, \ldots, n\}.$$

As the grid $0 = t_0 < t_1 < \ldots < t_n = T$ becomes finer and finer, what SDE will the above scheme become an approximative numerical solution of? **(5 points)**

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Solutions to Written Exam 23 August 2019

Task 1. To know the values of $[f{+}g,f{+}g](t)$ and $[f{-}g,f{-}g](t)$ is equivalent to know the values of $\frac{1}{4}\big([f{+}g,f{+}g](t)-[f{-}g,f{-}g](t)\big)=[f,g](t)$ and $\frac{1}{2}\big([f{+}g,f{+}g](t)+[f{-}g,f{-}g](t)\big)$ $=[f,f](t)+[g,g](t)$. Hence we can say what is the value of $[f,g](t)$, but not what is the values of $[f,f](t)$ or $[g,g](t)$ (as we know the value of the sum of these two only).

Task 2. The solution X is the stochastic exponential of the martingale $\{\int_0^t \sin(B(s))\, dB(s)\}_{t\geq 0}$, which in turn is given by $\{\exp[\int_0^t \sin(B(s))\,dB(s)-\frac{1}{2}\int_0^t \sin(B(s))^2\,ds]\}_{t\geq 0}$.

Task 3. The PDE is the Kolmogorov forward equation for the SDE $dX(t)=dt+dB(t)$, so that p is the transition density function for Brownian motion with unit drift, that is,

$$p(y,t)=\frac{1}{\sqrt{2\pi\,(t-s)}}\,\exp\bigg\{-\frac{(y-x-t+s)^2}{2\,(t-s)}\bigg\}\quad\text{for any }\;x,y\in\mathbb{R}\;\text{ and }\;0<s<t.$$

Task 4. By definition we have $\mathbf{E}\{I_A\,\mathbf{E}\{X\,|\,\mathcal{G}\}\}=\mathbf{E}\{I_A\,X\}$ for any event $A\in\mathcal{G}$. Taking $A=\Omega$, so that $I_A(\omega)=1$ for all $\omega\in\Omega$, we get $\mathbf{E}\{\mathbf{E}\{X\,|\,\mathcal{G}\}\}=\mathbf{E}\{X\}$. By definition $\mathbf{E}\{X\,|\,\{\emptyset,\Omega\}\}$ is the unique $\{\emptyset,\Omega\}$-measurable random variable that satisfies $\mathbf{E}\{I_A\,\mathbf{E}\{X\,|\,\{\emptyset,\Omega\}\}\}=\mathbf{E}\{I_A\,X\}$ for $A\in\{\emptyset,\Omega\}$. It follows that $\mathbf{E}\{X\,|\,\{\emptyset,\Omega\}\}=\mathbf{E}\{X\}$ as $\mathbf{E}\{X\}$ is $\{\emptyset,\Omega\}$-measurable (as is any non-random constant) and $\mathbf{E}\{I_A\,\mathbf{E}\{X\}\}$ $=\mathbf{E}\{I_A\,X\}$ holds trivially for $A=\emptyset$ and $A=\Omega$ with values 0 and $\mathbf{E}\{X\}$, respectively.

Task 5. For $\mu=0$ we see that the stationary distribution is the constant value zero. For $\mu>0$ the stationary distribution has probability density function $\pi(x)$ given by Equation 6.69 in Klebaner's book as $\pi(x)=C\,\exp\{\int_\infty^x 2\,\mu(y)/\sigma(y)^2\,dy\}/\sigma^2(x)=C\,\mathrm{e}^{-2\mu/(\sigma^2x)}/(\sigma^2x^2)=2\,\mu\,\mathrm{e}^{-2\mu/(\sigma^2x)}/(\sigma^2x^2)$ for $x>0$ and $\pi(x)=0$ otherwise, while similary for $\mu<0$ we have $\pi(x)=2\,\mu\,\mathrm{e}^{-2\mu/(\sigma^2x)}/(\sigma^2x^2)$ for $x<0$ and $\pi(x)=0$ otherwise.

Task 6. The scheme gives an approximate solution to the SDE $dY(t)=-Y(t)\,dt+Y(t)\,dB(t)+dY(t)\,dB(t)=-Y(t)\,dt+Y(t)\,dB(t)+\big(-Y(t)\,dt+Y(t)\,dB(t)+dY(t)\,dB(t)\big)$ $dB(t)=Y(t)\,dB(t)$ for $t\in[0,T]$ with initial value $Y(0)=1$, that is to say, an approximation of the stochastic exponential of B.