TMS165/MSA350 Stochastic Calculus

Written Exam Tuesday 29 October 2019 8.30 – 12.30

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Aids: Two sheets (=four pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed).

Grades: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

Motivations: All answers/solutions must be motivated. Good Luck!

Througout this exam $B = \{B(t)\}_{t\geq 0}$ denotes a Brownian motion.

Task 1. Find the covariation $[B_1, B_2](t)$ between two independent BM's $\{B_1(t)\}_{t\geq 0}$ and $\{B_2(t)\}_{t\geq 0}$. **(5 points)**

Task 2. Let $X(t)$ solve the SDE $dX(t) = \mu(X(t))\,dt + \sigma(X(t))\,dB(t)$ where $\sigma : \mathbb{R} \to (0,\infty)$ is continuously differentiable. It is possible to find a function $f : \mathbb{R} \to \mathbb{R}$ such that $Y(t) = f(X(t))$ satisfies the SDE $dY(t) = \hat{\mu}(Y(t))\,dt + dB(t)$ for some new drift coefficient function $\hat{\mu} : \mathbb{R} \to \mathbb{R}$. Find f and $\hat{\mu}$. **(5 points)**

Task 3. Find the Stratonovich stochastic exponential $X(t)$ of $B(t)$ given by $dX(t) = X(t)\,\partial B(t)$, $X(0) = 1$. **(5 points)**

Task 4. Find a solution to the PDE $\frac{\partial f(x,t)}{\partial t} + \frac{1}{2}\,\frac{\partial^2 f(x,t)}{\partial x^2} + x\,\frac{\partial f(x,t)}{\partial x} = f(x,t)$ for $t \in [0,T]$ with $f(x,T) = x$. **(5 points)**

Task 5. Let X be an $\mathrm{N}(0,1)$-distributed random variable on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Can X have unit $\mathbf{Q}$-mean Poisson distribution after a change of probability measure to $\mathbf{Q}(A) = \int_A \Lambda\,d\mathbf{P}$ for $A \in \mathcal{F}$ where $\Lambda \geq 0$ is unit $\mathbf{P}$-mean? **(5 points)**

Task 6. Explain the concept of Itô-Taylor expansion of a solution $X(t)$ to a time homogeneous SDE $dX(t) = \mu(X(t))\,dt + \sigma(X(t))\,dB(t)$. **(5 points)**

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Solutions to Written Exam 29 October 2019

Task 1. As $\{\frac{1}{\sqrt{2}}(B_1(t)+B_2(t))\}_{t\geq 0}$ and $\{\frac{1}{\sqrt{2}}(B_1(t)-B_2(t))\}_{t\geq 0}$ are also BM's we get $[B_1,B_2](t)=\frac{1}{4}\big([B_1+B_2,B_1+B_2](t)-[B_1-B_2,B_1-B_2](t)\big)=\frac{1}{2}\big([\frac{1}{\sqrt{2}}(B_1+B_2),\frac{1}{\sqrt{2}}(B_1+B_2)](t)-[\frac{1}{\sqrt{2}}(B_1-B_2),\frac{1}{\sqrt{2}}(B_1-B_2)](t)\big)=\frac{1}{2}\,(t-t)=0$.

Task 2. As $df(X(t))=f'(X(t))\,(\mu(X(t))\,dt+\sigma(X(t))\,dB(t))+\frac{1}{2}\,f''(X(t))\,\sigma(X(t))^2\,dt$ gives $f'(x)\,\sigma(x)=1$ we take $f(x)=\int_0^x\sigma(y)^{-1}\,dy$ with $\hat{\mu}(Y(t))=f'(X(t))\,\mu(X(t))+\frac{1}{2}\,f''(X(t))\,\sigma(X(t))^2=\mu(X(t))/\sigma(X(t))-\frac{1}{2}\,\sigma'(X(t))=\mu(f^{-1}(Y(t)))/\sigma(f^{-1}(Y(t)))-\frac{1}{2}\,\sigma'(f^{-1}(Y(t)))$.

Task 3. As $dX(t)=X(t)\,\partial B(t)=\frac{1}{2}\,X(t)\,dt+X(t)\,dB(t)$ we have $X(t)=\mathcal{E}(\frac{1}{2}\,t+B(t))$ $=\mathrm{e}^{\frac{1}{2}t+B(t)-\frac{1}{2}[\frac{1}{2}t+B(t),\frac{1}{2}t+B(t)]}=\mathrm{e}^{B(t)}$.

Task 4. By Feynman-Kac's formula $f(x,t)=\mathbf{E}\{\mathrm{e}^{-(T-t)}X(T)\,|\,X(t)=x\}$ where $X(t)$ solves the Langevin SDE $dX(t)=X(t)\,dt+dB(t)$ so that $X(T)=\mathrm{e}^T\big(X(0)+\int_0^T\mathrm{e}^{-s}\,dB(s)\big)=\mathrm{e}^T\big(\mathrm{e}^{-t}X(t)+\int_t^T\mathrm{e}^{-s}\,dB(s)\big)$ giving $f(x,t)=\ldots=x$. :)

Task 5. No as $\mathbf{Q}<<\mathbf{P}$ means $\mathbf{P}\{X=x\}=0$ implies $\mathbf{Q}\{X=x\}=0$ for all $x\in\mathbb{R}$.

Task 6. See Stig's lecture notes.