TMS165/MSA350 Stochastic Calculus

Written Exam Friday 3 January 2020 2 PM–6 PM

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AIDS: Two sheets (=four pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed).

GRADES: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Througout this exam $B = \{B(t)\}_{t \geq 0}$ denotes a Brownian motion.

Task 1. Consider the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with $\Omega = \{1, 2, 3\}$, $\mathcal{F}$ all subsets of Ω and $\mathbf{P}$ the uniform probability measure $\mathbf{P}(\{\omega\}) = \frac{1}{3}$ for $\omega \in \Omega$. Find $\mathbf{E}\{X|\mathcal{G}\}$ for the random variable $X(\omega) = \omega$ and σ-field $\mathcal{G} = \big\{\emptyset, \{1\}, \{2, 3\}, \Omega\big\}$. **(5 points)**

Task 2. Does the stochastic exponential of BM $\mathcal{E}(B(t))$ have a stationary PDF (/probability density function)? (Remember that answers must be motivated!) **(5 points)**

Task 3. You are given observed values $\{X(t)\}_{t \in [0,T]}$ of the solution to a Langevin SDE $dX(t) = -\alpha\, X(t)\, dt + \sigma\, dB(t)$ for $t \in [0, T]$ with unkonwn parameters $\alpha \in \mathbb{R}$ and $\sigma > 0$. How can the parameters be estimated using the observations? **(5 points)**

Task 4. Is it possible for $\{g(t)B(t)\, \mathrm{e}^{B(t)}\}_{t \in [0,T]}$ to be a martingale with respect to $\{\mathcal{F}_t^B\}_{t \in [0,T]}$ for some choice of non-random continuous function $g : [0, T] \to \mathbb{R}$ that is not zero for all $t \in [0, T]$? **(5 points)**

Task 5. Consider a time homogeneous SDE $dX(t) = \mu(X(t))\, dt + \sigma(X(t))\, dB(t)$ for $t \in [0, T]$, $X(0) = x_0$, with continuous coefficient functions $\mu, \sigma : \mathbb{R} \to \mathbb{R}$. Give necessary and sufficient conditions for the solution $\{X(t)\}_{t \in [0,T]}$ to have zero quadratic variation $[X, X](t) = 0$ for $t \in [0, T]$. **(5 points)**

Task 6. Derive the Milstein method for approximative numerical solution of SDE.

(5 points)

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Solutions to Written Exam 3 January 2020

Task 1. $\mathbf{E}\{X|\mathcal{G}\}(\omega) = 1$ for $\omega = 1$ and $\frac{5}{2}$ for $\omega \in \{2,3\}$ as $\mathbf{E}\{1 \cdot \mathbf{1}_{\{1\}}(\omega)\} = 1 \cdot \frac{1}{3} = \mathbf{E}\{X\,\mathbf{1}_{\{1\}}(\omega)\}$ and $\mathbf{E}\{\frac{5}{2} \cdot \mathbf{1}_{\{2,3\}}(\omega)\} = \frac{5}{2} \cdot \frac{2}{3} = \mathbf{E}\{X\,\mathbf{1}_{\{2,3\}}(\omega)\}$.

Task 2. As $\mathcal{E}(B(t))$ is the solution to the time homogeneous SDE $dX(t) = X(t)\,dB(t)$ with $\mu(x) = 0$ and $\sigma(x) = x$ a stationary PDF π must be given by

$$\pi(x) = \frac{C}{\sigma(x)^2}\,\exp\biggl\{\int_{x_0}^{x} \frac{2\,\mu(y)}{\sigma(y)^2}\,dy\biggr\} = \frac{C}{x^2},$$

where $C > 0$ must be selected to make $\int_{-\infty}^{\infty} \pi(x)\,dx = 1$. However, the proposed π is not integrable at zero and so there exists no stationary PDF.

Task 3. As $[X,X](t) = \int_0^t \sigma(X(s))^2\,ds = \sigma^2 t$ we take $\hat{\sigma}^2 = \frac{1}{T}\sum_{i=1}^{n}(X(t_i) - X(t_{i-1}))^2$ for a sufficiently fine grid $0 = t_0 < t_1 < \ldots < t_n = T$. The estimate of α is Eq. 10.55 in Klebaner's book.

Task 4. By Itô's formula we have

$$d\bigl(g(t)B(t)\,\mathrm{e}^{B(t)}\bigr) = \bigl(g'(t)B(t) + g(t) + \tfrac{1}{2}g(t)B(t)\bigr)\,\mathrm{e}^{B(t)}\,dt + \bigl(g(t) + g(t)B(t)\bigr)\,\mathrm{e}^{B(t)}\,dB(t).$$

For a martingale we must therefore have either $g(t) = 0$ or $\frac{1}{2} + g'(t)/g(t) = -1/B(t)$ making $g(t) = \exp\{-\frac{1}{2}t - \int 1/B(t)\,dt\}$ random. So the answer is no!

Task 5. As $[X,X](T) = \int_0^T \sigma(X(t))^2\,dt = 0$ implies $\sigma(X(t)) = 0$ for $t \in [0,T]$ (as $\sigma(X(\cdot))$ is continuous) so that $X(t) = x_0 + \int_0^t \mu(X(s))\,ds$ and the SDE is an ODE. Conversely the solution to an ODE has zero quadratic variation [as $X(\cdot)$ is continuously differentiable being the integral of continuous $\mu(X(\cdot))$].

Task 6. See Stig Larsson's lecture notes.