TMS165/MSA350 Stochastic Calculus
Written home exam Friday 21 August 2020 8.30–12.30

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AIDS: All aids are permitted. (See the Canvas course "TMS165 Re-Exam TMS165/ MSA350" with instructions for this reexam for clarifications.)

GRADES: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Througout this exam $B = \{B(t)\}_{t\geq 0}$ denotes a Brownian motion.

Task 1. Let (X, Y) be a continuous two-dimensional random variable with PDF (/probability density function) $f_{X,Y}(x, y)$ such that $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx > 0$ for $y \in \mathbb{R}$. Show that $\mathbf{E}\{X|\sigma(Y)\} = g(Y)$ where $g(y) = \mathbf{E}\{X|Y = y\} = \int_{-\infty}^{\infty} x\, f_{X,Y}(x, y)/f_Y(y)\, dx$. [HINT: $\sigma(Y) = \{\mathbf{1}_A(Y) : A \in \mathcal{F}\}$.] **(5 points)**

Task 2. Find the PDE that is satisfied for functions $f(x, t)$ that make $\{f(B(t), t) - B(t)^4\}_{t\geq 0}$ a martingale. Also, give an example of such a function $f(x, t)$. **(5 points)**

Task 3. Let $X(t)$ solve the SDE $dX(t) = \mu(X(t))\, dt + \sigma(X(t))\, dB(t)$ where $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to (0, \infty)$ are continuous. It is possible to find a function $f : \mathbb{R} \to \mathbb{R}$ such that $Y(t) = f(X(t))$ satisfies the SDE $dY(t) = \hat{\sigma}(Y(t))\, dB(t)$ for some new diffusion coefficient function $\hat{\sigma} : \mathbb{R} \to (0, \infty)$. Find f and $\hat{\sigma}$. **(5 points)**

Task 4. Given constants $\mu, \sigma \in \mathbb{R}$, find the solution to the PDE

$$\frac{\partial f}{\partial t} + \mu x \frac{\partial f}{\partial x} + \frac{\sigma^2 x^2}{2} \frac{\partial^2 f}{\partial x^2} = f(x, t) \text{ for } t \in [0, T] \text{ and } x > 0, \quad f(x, T) = \ln(x).$$ **(5 points)**

Task 5. Consider the SDE $dX(t) = -\alpha\, X(t)\, dt + \sigma\, dB(t)$ for $t \geq 0$, for some constants $\alpha \in \mathbb{R}$ and $\sigma > 0$. If you are given an observation $\{X(t)\}_{t\in[0,T]}$ of the solution, how can it be used to estimate α and σ (when you do not know their values)? **(5 points)**

Task 6. For a general SDE $dX(t) = \mu(X(t), t)\, dt + \sigma(X(t), t)\, dB(t)$, $t \in [0, T]$, with $X(0) = x_0$, consider the half implicit Euler scheme $\hat{Y}(t_0) = x_0$ and

$$\hat{Y}(t_i) = \hat{Y}(t_{i-1}) + \frac{\mu(\hat{Y}(t_{i-1}), t_{i-1}) + \mu(\hat{Y}(t_i), t_i)}{2} (t_i - t_{i-1}) + \frac{\sigma(\hat{Y}(t_{i-1}), t_{i-1}) + \sigma(\hat{Y}(t_i), t_i)}{2} (B(t_i) - B(t_{i-1}))$$

for $i = 1, \ldots, n$, with $0 = t_0 < t_1 < \ldots < t_n = T$. For which SDE $dY(t) = \hat{\mu}(Y(t), t)\, dt + \hat{\sigma}(Y(t), t)\, dB(t)$ is $\{\hat{Y}(t_i)\}_{i=0}^n$ a numerical approximation of $\{Y(t_i)\}_{i=0}^n$? **(5 points)**

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Solutions to Written Exam 21 August 2020

Task 1. $\mathbf{E}\big\{\mathbf{1}_A(Y)[\int_{-\infty}^{\infty} x\, f_{X,Y}(x,Y)/f_Y(Y)\,dx]\big\} = \int_{-\infty}^{\infty} \mathbf{1}_A(y)[\int_{-\infty}^{\infty} x\, f_{X,Y}(x,y)\,dx]\,dy = \mathbf{E}\{\mathbf{1}_A(Y)X\}$ for $A \in \mathcal{F}$.

Task 2. According to Corollary 6.4 in Klebaner's book $g(X(t),t) = f(X(t),t) - X(t)^4$ with $X(t) = B(t)$ is a martingale when $L_t\, g(x,t) + \frac{\partial g}{\partial t} = \frac{1}{2}\frac{\partial^2 g}{\partial x^2} + \frac{\partial g}{\partial t} = \frac{1}{2}\frac{\partial^2 f}{\partial x^2} - 6x^2 + \frac{\partial f}{\partial t}$ $= 0$. This PDE is satisfied, e.g., for $f(x,t) = x^4$ giving $X(t) = B(t)^4$. (But there are also other more complicated solutions.)

Task 3. As $df(X(t)) = f'(X(t))\,(\mu(X(t))\,dt + \sigma(X(t))\,dB(t)) + \frac{1}{2}\,f''(X(t))\,\sigma(X(t))^2\,dt$ gives $f'(x)\,\mu(x) + \frac{1}{2}f''(x)\,\sigma(x)^2 = 0$ we take $f(x) = \int \exp\{-\int_{x_0}^{x} 2\,\mu(y)/\sigma(y)^2\,dy\}\,dx$ and $\hat{\sigma}(Y(t)) = f'(X(t))\,\sigma(X(t)) = \exp\{-\int_{x_0}^{f^{-1}(Y(t))} 2\,\mu(y)/\sigma(y)^2\,dy\}\,\sigma(f^{-1}(Y(t)))$.

Task 4. With $X(t) = x_0\,\mathrm{e}^{(\mu-\frac{1}{2}\sigma^2)t+\sigma B(t)}$ the solution to the SDE $dX(t) = \mu\,X(t)\,dt + \sigma\,X(t)\,dB(t)$, $X(t) = x_0$, Feynman-Kac formula gives

$$f(x,t) = \mathbf{E}\big\{\mathrm{e}^{-(T-t)}\ln(X(T))\;\big|\;X(t) = x\big\} = \mathrm{e}^{-(T-t)}\big(\ln(x) + (\mu - \tfrac{1}{2}\sigma^2)(T-t)\big).$$

Task 5. See Example 10.5 in Klebaner's book as well as Patrik's lecture notes on applications for estimation of α. As for estimation of σ we have $[X,X](t) = \sigma^2 t$ so that $\sum_{i=1}^{n}(X(t_i) - X(t_{i-1}))^2 \approx \sigma^2 T$ for a tight grid $0 = t_0 < t_1 < \ldots < t_n = T$, giving $\hat{\sigma}^2 = \frac{1}{T}\sum_{i=1}^{n}(X(t_i) - X(t_{i-1}))^2$.

Task 6. By basic stochastic calculus the half implicit scheme adds $\frac{1}{2}\int_0^t \sigma(Y(s),s) \times \sigma'(Y(s),s)\,ds$ to the solution as compared with the "unbiased" explicit Euler scheme. Hence the half implicit scheme is a numerical approximation of the solution to the SDE

$$dY(t) = \big(\mu(Y(t),t) + \tfrac{1}{2}\,\sigma(Y(t),t)\sigma'(Y(t),t)\big)\,dt + \sigma(Y(t),t)\,dB(t).$$