TMS 165/MSA350 Stochastic Calculus

Home Exercises for Chapters 1-2 in Klebaner's Book

Task 1. Find a function with infinite variation over a finite interval.

Task 2. Express $[f, g]$ in terms of $[f+g]$ and $[f-g]$.

Task 3. Explain why the Riemann-Stieltjes integral $\int f\,dg$ defined through Equation 1.19 in Klebaner's book is not well-defined when $[f, g] \neq 0$.

Task 4. For the function $f : [0,1] \to \{0,1\}$ given by $f(x) = 0$ for $x \in [0,1] \setminus \mathbb{Q}$ and $f(x) = 1$ for $x \in [0,1] \cap \mathbb{Q}$ we have $\{x \in [0,1] : f(x) = 1\} \subseteq \cup_{i=1}^{\infty} [q_i - 2^{-i}\varepsilon, q_i + 2^{-i}\varepsilon] \equiv L(\varepsilon)$ for each $\varepsilon > 0$ and any enumeration $\{q_i\}_{i=1}^{\infty}$ of $[0,1] \cap \mathbb{Q}$, where the length of $L(\varepsilon)$ is less or equal ε. Find the value of the Lebesgue integral $\int_0^1 f(x)\,dx$.

Task 5. Prove Equation 2.19 in Klebaner's book.

Task 6. Find $\mathbf{E}\{X \,|\, \sigma(Y)\}$ for a standardized (to have zero mean and unit variance) bivariate normal distributed random variable (X, Y) such that X and Y has correlation $\rho \in (-1, 1)$.

$\boxed{\text{Task 1}}$ Consider $V_g([0,1])$ for the function $g : [0,1] \to [-1,1]$ given by

$$g(x) = \begin{cases} \sin(1/x), & x \in (0,1] \\ 0, & x = 0 \end{cases}.$$

Clearly $V_g([0,1]) = +\infty$.

$\boxed{\text{Task 2}}$ $\frac{1}{4}([f+g] - [f-g]) = \frac{1}{4}([f+g, f+g] - [f-g, f-g])$

$= \frac{1}{4}([f,f] + [f,g] + [g,f] + [g,g] - [f,f] + [f,g] + [g,f] - [g,g])$

$= [f,g]$

$\boxed{\text{Task 3}}$ $\lim_{\max_{1 \le i \le n} t_i - t_{i-1} \to 0} \sum_{i=1}^{n} (f(t_i) - f(t_{i-1}))(g(t_i) - g(t_{i-1})) \neq 0 \Rightarrow$

$\sum_{i=1}^{n} f(t_i)(g(t_i) - g(t_{i-1}))$ and $\sum_{i=1}^{n} f(t_{i-1})(g(t_i) - g(t_{i-1}))$

converge to different values (if they converge at all) as $\max_{1\le i\le n} t_i - t_{i-1} \to 0$ which in turn means that Riemann-Stieltjes integral $\int f\,dg$ is not well-defined.

$\boxed{\text{Task 4}}$ As the Lebesgue integral is an approximation converging from below it is easy to see that

$$0 \le \text{Lebesgue-integral } \int_0^1 f(x)\,dx$$

$$\le 1 \times \text{length}\{x\in[0,1] : f(x)=1\}$$

$$+\, 0 \times \text{length}\{x\in[0,1] : f(x)=0\}$$

$$= \text{length}\{[0,1]\cap\mathbb{Q}\}$$

$$\le \text{length}\{L(\varepsilon)\} \qquad (\text{as } [0,1]\cap\mathbb{Q}\subseteq L(\varepsilon))$$

$$\le \sum_{i=1}^{+\infty} 2\cdot 2^{-i}\,\varepsilon = 2\varepsilon \quad \text{for each } \varepsilon>0$$

which means that $\int_0^1 f(x)\,dx$ must be 0.

$\boxed{\text{Task 5}}$ The definition of $E(X|\mathcal{G})$ is that this r.v. should be $\mathcal{G}$-measurable with $E(1_B E(X|\mathcal{G})) = E(1_B X)$ for $B\in\mathcal{G}$.

This means that $E(E(X|\mathcal{G}_2)|\mathcal{G}_1)$ should be $\mathcal{G}_1$-measurable with $E(1_B E(E(X|\mathcal{G}_2)|\mathcal{G}_1)) = E(1_B E(X|\mathcal{G}_2))$ for $B\in\mathcal{G}_1$. So let us check $E(X|\mathcal{G}_1)$ fixes that:

$$E(1_B E(X|\mathcal{G}_1)) \underset{\substack{\uparrow \\ B\in\mathcal{G}_1}}{=} E(1_B X) \underset{\substack{\uparrow \\ B\in\mathcal{G}_1\subseteq\mathcal{G}_2}}{=} E(1_B E(X|\mathcal{G}_2))$$

and as $E(X|\mathcal{G}_1)$ is $\mathcal{G}_1$-measurable we have it!

$\boxed{\text{Task 6}}$ As $Cov(X-\rho Y, Y) = Cov(X,Y) - \rho\, Cov(Y,Y) = \rho - \rho = 0$ the r.v.'s $X-\rho Y$ and Y are independent, so that

$$E(X|\sigma(Y)) = E(X-\rho Y|\sigma(Y)) + E(\rho Y|\sigma(Y))$$

$$= E(X-\rho Y) + \rho Y = \rho Y.$$