Home Exercises for Chapter 10 in Klebaner's Book

Task 5. Can an Ornstein-Uhlenbeck process [a solution to the Langevin equation 5.12 in Klebaner's book] become a Brownian motion by means of a change of measure? In that case, how? Otherwise, why not?

Task 6. Assume that we have observed the solution $\{X(t)\}_{t\in[0,T]}$ to the CKLS SDE in Task 1 where the γ coefficient is known. Find the estimators of the coefficients α, β and σ according to the methodolgy of Section 10.6 in Klebaner's book.

Solutions to Home Exercises on Chapter 10

Task 5 If $dX(t) = -\alpha X(t)dt + \sigma\, dB(t)$ is a P-Ornstein Uhlenbeck process then by the subsection "Change of Drift in Diffusions" (page 280) together with Eq. 10.52 in Klebaner's book $X(t)$ satisfies $dX(t) = \sigma\, dW(t)$ for $W(t)$ a Q-BM when

$$\frac{dQ}{dP} = \exp\left(\int_0^T \frac{\alpha X(t)}{\sigma}\, dX(t) - \frac{1}{2}\int_0^T \frac{\alpha^2 X(t)^2}{\sigma^2}\, dt\right).$$

Hence $X(t)$ can be made a BM by change of measure for $\sigma = 1$ but not otherwise. [Of course, $\frac{1}{\sigma}X(t)$ will be Q-BM but this is another process and not the some process just considered under antoher probability.]

Task 6 As $[X,X](t) = \int_0^t \sigma^2 X(s)^{2\gamma}\, ds$ we can estimate $\hat{\sigma}^2$ with $[X,X]'(T)/X(T)^{2\gamma}$. By the subsection "Likelihood Ratios for Diffusions" (pages 285-286) in Klebaner's book we further have

$$\frac{dP_{\alpha,\beta}}{dP_{0,0}} = \exp\left(\int_0^T \frac{\alpha + \beta X(t)}{\sigma^2 X(t)^{2\gamma}}\, dX(t) - \frac{1}{2}\int_0^T \frac{(\alpha + \beta X(t))^2}{\sigma^2 X(t)^{2\gamma}}\, dt\right)$$

$$\frac{d}{d\alpha}\frac{P_{\alpha,\beta}}{P_{0,0}} = \left(\int_0^T \frac{dX(t)}{\sigma^2 X(t)^{2\gamma}} - \int_0^T \frac{\alpha + \beta X(t)}{\sigma^2 X(t)^{2\gamma}}\, dt\right)\exp(\ldots)$$

$$\frac{d}{d\beta}\frac{P_{\alpha,\beta}}{P_{0,0}} = \left(\int_0^T \frac{X(t)\, dX(t)}{\sigma^2 X(t)^{2\gamma}} - \int_0^T \frac{X(t)(\alpha + \beta X(t))}{\sigma^2 X(t)^{2\gamma}}\, dt\right)\exp(\ldots)$$

which we can set both to zero and solve for α and β in order toget the maximum likelihood ratio estimates of these parameters.