TMS 165/MSA350 Stochastic Calculus

Home Exercises for Chapter 3 in Klebaner's Book

Througout this set of exercises $B = \{B(t)\}_{t\geq 0}$ denotes Brownian motion.

Task 1. Show that the stochastic process $\{B(t)^4 - 6tB(t)^2 + 3t^2\}_{t\geq 0}$ is a martingal with respect to the filtration $\{\mathcal{F}_t^B\}_{t\geq 0}$ generated by B.

Task 2. For an $\varepsilon > 0$, consider the differential ratio process $\Delta_\varepsilon = \{\Delta_\varepsilon(t)\}_{t\geq 0}$ given by

$$\Delta_\varepsilon(t) = \frac{B(t+\varepsilon) - B(t)}{\varepsilon} \quad \text{for } t \geq 0.$$

Show that the covariance function

$$r_\varepsilon(t) = \mathbf{Cov}\{\Delta_\varepsilon(s), \Delta_\varepsilon(s+t)\}$$

of Δ_ε is a triangle like function that depends on the difference t between $s \geq 0$ and $s+t \geq 0$ only. Show that $r_\varepsilon(t) \to \delta(t)$ (Dirac's δ-function) as $\varepsilon \downarrow 0$. Simulate a sample path of $\{\Delta_\varepsilon(t)\}_{t\in[0,1]}$ for a really small $\varepsilon > 0$ and plot it graphically. Discuss the claim that the (non-existing in the usual sense) derivative process $\{B'(t)\}_{t\geq 0}$ of B is white noise.

Task 3. Nobert Wiener (1894-1964) defined the stochastic integral process $\{\int_0^t g\,dB\}_{t\geq 0}$ with respect to B for continuously differentiable functions $g : [0,\infty) \to \mathbb{R}$ as

$$\int_0^t g\,dB = g(t)B(t) - \int_0^t B\,dg = g(t)B(t) - \int_0^t B(r)g'(r)\,dr \quad \text{for } t \geq 0.$$

[Of course, the motivation for this definition comes from the integration by parts formula Equation 1.20 in Klebaner's book.] Show by means of direct calculation (not using Itô's formula) that $\{\int_0^t g\,dB\}_{t\geq 0}$ defined in this way is a martingale.

Task 4. As B has strictly positive quadratic variation and is continuous, B must have infinite variation V_B by Theorem 1.10 in Klebaner's book. Another way to understand that $V_B(t) = \infty$ for $t > 0$ is the following: For increasingly fine partitions $0 = t_0 < t_1 < \ldots < t_n = t$ of the interval $[0,t]$, compute the limits of

$$\mathbf{E}\left\{\sum_{i=1}^n |B(t_i) - B(t_{i-1})|\right\} \quad \text{and} \quad \mathbf{Var}\left\{\sum_{i=1}^n |B(t_i) - B(t_{i-1})|\right\}$$

as $\max_{1\leq i\leq n} t_i - t_{i-1} \downarrow 0$. Explain how to conclude that $V_B(t) = \infty$.

$\boxed{\text{Task 1}}$

$$E(B(t)^4 - 6tB(t)^2 + 3t^2 \mid \mathcal{F}_s^B) = E((B(t)-B(s))^4 + 4(B(t)-B(s))^3 B(s) + 6(B(t)-B(s))^2 B(s)^2 + 4(B(t)-B(s))B(s)^3$$
$$+ B(s)^4 - 6t(B(t)-B(s))^2 - 12t(B(t)-B(s))B(s) + 6tB(s)^2 + 3t^2 \mid \mathcal{F}_s^B) = 3(t-s)^2 + \cancel{4\cdot 0\cdot B(s)}$$
$$+ 6(t-s)B(s)^2 + \cancel{4\cdot 0\cdot B(s)^3} + B(s)^4 - 6t(t-s) - \cancel{12t\cdot 0\cdot B(s)} - 6tB(s)^2 + 3t^2 =$$

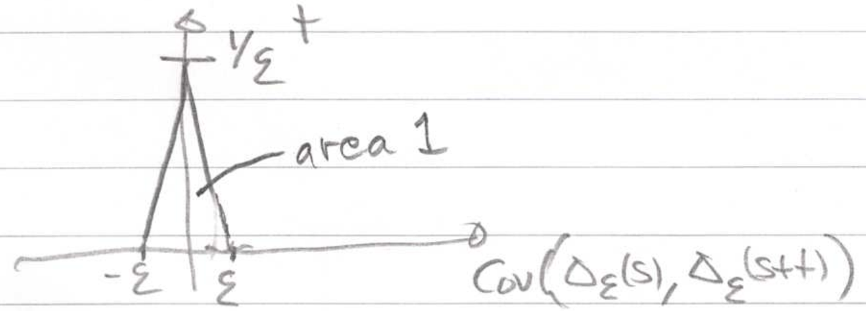
$$= \ldots = B(s)^4 - 6sB(s)^2 + 3s^2 \text{ for } 0 \le s \le t.$$

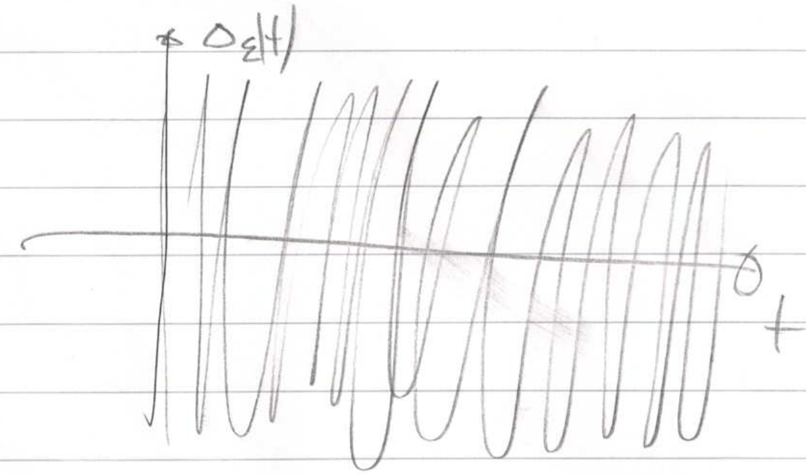
$\boxed{\text{Task 2}}$ $$\mathrm{Cov}(\Delta_\varepsilon(s), \Delta_\varepsilon(s+t)) = E\left(\frac{B(s+\varepsilon)-B(s)}{\varepsilon} \, \frac{B(s+t+\varepsilon)-B(s+t)}{\varepsilon} \right)$$

$$= \frac{\min(s+\varepsilon, s+t+\varepsilon) - \min(s+\varepsilon, s+t) - \min(s, s+t+\varepsilon) + \min(s, s+t)}{\varepsilon^2}$$

$$= \frac{1}{\varepsilon^2}\left(\min(\varepsilon, t+\varepsilon) - \min(\varepsilon, t) - \min(0, t+\varepsilon) + \min(0, t) \right)$$

$$= \frac{1}{\varepsilon^2} \begin{cases} \varepsilon - \varepsilon - 0 - 0 = 0 & \text{for } t \ge \varepsilon \\ \varepsilon - t - 0 - 0 = \varepsilon - t & \text{for } 0 \le t \le \varepsilon \\ \varepsilon + t - t - 0 - t = \varepsilon + t & \text{for } -\varepsilon \le t \le 0 \\ t + \varepsilon - t - (t+\varepsilon) - t = 0 & \text{for } t \le -\varepsilon \end{cases}$$





$\mathrm{Cov}(\Delta_\varepsilon(s), \Delta_\varepsilon(s+t)) \to \delta(t)$ as $\varepsilon \downarrow 0$ which is what it should be for white noise.

$\boxed{\text{Task 3}}$ $E\left(\int_0^t g\,dB \,\middle|\, \mathcal{F}_s^B\right) = E\left(g(t)B(t) - \int_0^t B(r)g'(r)dr \,\middle|\, \mathcal{F}_s^B\right)$

$= g(t)B(s) - E\left(\int_s^t (B(r)-B(s))g'(r)dr \,\middle|\, \mathcal{F}_s^B\right) - E\left(\int_s^t B(s)g'(r)dr \,\middle|\, \mathcal{F}_s^B\right)$

$- E\left(\int_0^s B(r)g'(r)dr \,\middle|\, \mathcal{F}_s^B\right)$

$= g(t)B(s) - 0 - B(s)(g(t)-g(s)) - \int_0^s B(r)g'(r)dr$

$= g(s)B(s) - \int_0^s B(r)g'(r)dr = \int_0^s g\,dB$ for $0 \le s \le t$.

$\boxed{\text{Task 4}}$ $E\left(\sum_{i=1}^n |B(t_i)-B(t_{i-1})|\right) = \sum_{i=1}^n E(|B(t_i)-B(t_{i-1})|)$

$= \sum_{i=1}^n E(|N(0,t_i-t_{i-1})|) = \sum_{i=1}^n E(\sqrt{t_i-t_{i-1}}\,|N(0,1)|) \to \infty$

$\mathrm{Var}\left(\sum_{i=1}^n |B(t_i)-B(t_{i-1})|\right) = \sum_{i=1}^n \mathrm{Var}(|B(t_i)-B(t_{i-1})|)$

$= \sum_{i=1}^n \mathrm{Var}(\sqrt{t_i-t_{i-1}}\,|N(0,1)|) = t\,\mathrm{Var}(|N(0,1)|)$

as $\max_{1\le i\le n} t_i - t_{i-1} \searrow 0$.