TMS 165/MSA350 Stochastic Calculus

Home Exercises for Chapter 4 in Klebaner's Book

Througout this set of exercises $B = \{B(t)\}_{t\geq 0}$ denotes Brownian motion.

Task 1. Show that a sequence $\{X_n\}_{n=1}^{\infty}$ of random variables such that $\mathbf{E}\{X_n^2\} < \infty$ for all n converges in $\mathbb{L}^2$ to some random variable X if and only if the limit $\lim_{m,n\to\infty} \mathbf{E}\{X_m X_n\}$ exists.

Task 2. Show the isometry property Equation 4.12 in Klebaner's book for the Itô integral process $\{\int_0^t X\, dB\}_{t\in[0,T]}$ for $X \in E_T$, e.g., using that the property holds for $X \in S_T$ [cf. Equation 4.5 in Klebaner's book].

Task 3. Show that for an $X \in P_T$ we have in the sense of convergence in probability

$$\int_0^T (X_n(t) - X(t))^2\, dt \to 0 \quad \text{as } n\to\infty \quad \text{for some sequence } \{X_n\}_{n=1}^{\infty} \subseteq S_T,$$

and that the Itô integral process $\{\int_0^t X\, dB\}_{t\in[0,T]}$ is well-defined as a limit in the sense of convergence in probability of $\int_0^t X_n\, dB$ as $n\to\infty$ for $t\in[0,T]$.

Task 4. Show that for a process $X \in P_T$ we have

$$\mathbf{P}\left\{\int_0^T X(t)^2\, dt = 0\right\} = 1 \Leftrightarrow \mathbf{P}\left\{\int_0^t X\, dB = 0\right\} = 1 \quad \text{for } t\in[0,T].$$

Task 5. Find stochastic processes $\{X(t)\}_{t\in[0,1]}$, $\{Y(t)\}_{t\in[0,1]}$ and $\{Z(t)\}_{t\in[0,1]}$ that belong to E_1, $P_1\backslash E_1$ and P_1^c, respectively.

Task 6. Apply Itô's formula Theorem 4.17 in Klebaner's book to $f(X(t), Y(t))$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f(x,y) = g(x)\,y$ for a two times continuously differentiable function $g : \mathbb{R}\to\mathbb{R}$ and $X = Y = B$. Compare with what you get from applying the integration by parts formula Equation 4.57 in Klebaner's book with $X = g(B)$ and $Y = B$. Derive from the comaprison a new proof (without any explicit calculations other than applications of Itô's formula) of the property established in Example 4.23 in Klebaner's book that $[g(B), B](t) = \int_0^t g'(B(s))\, ds$ for $t\geq 0$.

$\boxed{\text{Task 1}}$ From the Cauchy Criterion we have $\mathbb{L}^2$-convergence iff $\lim_{m,n\to\infty} E((X_m - X_n)^2) = 0$.

$\boxed{\Leftarrow}$ $\lim_{m,n\to\infty} E(X_m X_n) = \varphi$ exists $\Rightarrow$ $0 = \varphi - 2\varphi + \varphi = \lim_{m\to\infty} E(X_m^2) - \lim_{m,n\to\infty} E(2X_m X_n) + \lim_{n\to\infty} E(X_n^2) = \lim_{m,n\to\infty} E((X_m - X_n)^2) \Rightarrow$ $\mathbb{L}^2$-convergence.

$\boxed{\Rightarrow}$ $\mathbb{L}^2$-convergence $X_n \to_{\mathbb{L}^2} X \Rightarrow \lim_{n\to\infty} E(X_n^2) = E(X^2)$ by the solved exercises to Chapter 4. Putting this together with the Cauchy Criterion we get $E(X_m X_n) = \frac{1}{2}E(X_m^2) + \frac{1}{2}E(X_n^2) - \frac{1}{2}E((X_m - X_n)^2) \to \frac{1}{2}E(X^2) + \frac{1}{2}E(X^2) - 0$ as $m,n \to \infty$.

$\boxed{\text{Task 2}}$ For $X \in E_T$ we have $\int_0^T X_n \, dB \to_{\mathbb{L}^2} \int_0^T X \, ds$ as $n \to \infty$ for $\{X_n\}_{n=1}^{+\infty} \subseteq S_T$ with $\int_0^T E((X_n(t) - X(t))^2)\,dt \to 0$. By the isometry property for $X_n \in S_T$ together with the fact that $Y_n \to_{\mathbb{L}^2} Y \Rightarrow E(Y_n^2) \to E(Y^2)$ (see above) it follows that $E((\int_0^T X\,dB)^2)$ $\leftarrow E((\int_0^T X_n\,dB)^2) = \int_0^T E(X_n(t)^2) = \int_0^T E((X_n(t) - X(t))^2)\,dt + 2\int_0^T E((X_n(t) - X(t))X(t))\,dt + \int_0^T E(X(t)^2)\,dt \to \int_0^T E(X(t)^2)\,dt$ since $\left|\int_0^T E((X_n(t) - X(t))X(t))\,dt\right| \le \int_0^T |E((X_n(t) - X(t))X(t))|\,dt$ $\le \int_0^T \sqrt{E((X_n(t) - X(t))^2)}\sqrt{E(X(t)^2)}\,dt$ $\le \sqrt{\int_0^T E((X_n(t) - X(t))^2)\,dt}\sqrt{\int_0^T E((X(t))^2)\,dt} \to 0$.

Task 3 It is enough to show that for each $n \in \mathbb{N}$ there exists $X_n \in S_T$ such that $P(\int_0^T (X_n(t)-X(t))^2 dt > \frac{1}{n}) < \frac{1}{n}$. Now take $Y_n \in E_T$ such that $P(\int_0^T (Y_n(t)-X(t))^2 dt > \frac{1}{4n}) < \frac{1}{2n}$ and $X_n \in S_T$ such that $E(\int_0^T (X_n(t)-Y_n(t))^2 dt) < \frac{1}{8n^2}$. Then $P(\int_0^T (X_n(t)-X(t))^2 dt > \frac{1}{n}) \leq P(2\int_0^T (X_n(t)-Y_n(t))^2 dt > \frac{1}{2n}) + P(2\int_0^T (Y_n(t)-X(t))^2 dt > \frac{1}{2n}) \leq E(\int_0^T (X_n(t)-Y_n(t))^2 dt)/\frac{1}{4n} + \frac{1}{2n} < \frac{1}{8n^2}/\frac{1}{4n} + \frac{1}{2n} = \frac{1}{n}$.

Task 4 $\boxed{\Rightarrow}$ If $P(\int_0^T X(t)^2 dt = 0) = 1$ we can take $X_n \equiv 0 \in S_T$ and have $\int_0^T (X_n(t)-X(t))^2 dt \to_P 0$ as $n \to \infty$ so that per definition of the Itô integral for P_T we have $\int_0^T X\, dB = 0$.

$\boxed{\Leftarrow}$ If $P(\int_0^t X\, dB = 0) = 1$ for $t \in [0,T]$, then by continuity of $\{\int_0^t X\, dB\}_{t\in[0,T]}$ we have $P(\int_0^t X\, dB = 0 \ \forall\, t \in [0,T]) = 1$ which implies that $[\int_0^T X\, dB, \int_0^T X\, dB] = \int_0^T X(t)^2 dt = 0$ wp 1.

Task 5 $\boxed{E_1}$ $B \in E_1$ since $\int_0^1 E(B(t)^2)\, dt = \int_0^1 t\, dt = 1/2$.

$\boxed{P_1 \setminus E_1}$ $e^{B^2} \in P_1 \setminus E_1$ since the process is continuous with $E(e^{B(t)^2}) = E(e^{N(0,t)^2}) = \int_{-\infty}^{+\infty} e^{x^2} \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx = \infty$ for $t \in [1/2, 1]$ making $\int_0^1 E(e^{B(t)^2}) dt = \infty$.

$\boxed{P_1^c}$ Any non-adapted to $\{\mathcal{F}_t^B\}_{t\geq 0}$ works fine [such as, e.g., $B(t)+\xi$ where ξ non-deterministic and independent of B] as well as, e.g., the (non-random) process $X(t) = (1/2-t)^{-1/2}$ for $t \in [0, 1/2)$, $X(1/2) = 0$, $X(t) = (t-1/2)^{-1/2}$ for $t \in (1/2, 1]$.

Task 6

By Itô's formula for $f(X(t), Y(t))$ with $f(x,y) = g(x)y$ and $X = Y = B$ we have $d\big(g(B(t))B(t)\big) = g'(B(t))B(t)\,dB(t) + g(B(t))\,dB(t) + \frac{1}{2}g''(B(t))B(t)\,dt + g'(B(t))\,dt$ while integration by parts for $g(B)$ and B gives $d\big(g(B(t))B(t)\big) = d\big(g(B(t))\big)B(t) + g(B(t))\,dB(t) + d[g(B(t)), B(t)] = g'(B(t))B(t)\,dB(t) + \frac{1}{2}g''(B(t))B(t)\,dt + g(B(t))\,dB(t) + d[g(B(t)), B(t)]$. Comparing these two results we conclude that $d[g(B(t)), B(t)] = g'(B(t))\,dt$.