TMS 165/MSA350 Stochastic Calculus

Home Exercises for Chapter 4 in Klebaner’s Book

Throughout this set of exercises $B = \{B(t)\}_{t \geq 0}$ denotes Brownian motion.

**Task 1.** Show that a sequence $\{X_n\}_{n=1}^\infty$ of random variables such that $\mathbf{E}(X_n^2) < \infty$ for all $n$ converges in $L^2$ to some random variable $X$ if and only if the limit $\lim_{m,n \to \infty} \mathbf{E}\{X_m X_n\}$ exists.

**Task 2.** Show the isometry property Equation 4.12 in Klebaner’s book for the Itô integral process $\{\int_0^t X dB\}_{t \in [0,T]}$ for $X \in E_T$, e.g., using that the property holds for $X \in S_T$ [cf. Equation 4.5 in Klebaner’s book].

**Task 3.** Show that for an $X \in P_T$ we have in the sense of convergence in probability

$$\int_0^T (X_n(t) - X(t))^2 \, dt \to 0 \quad \text{as} \quad n \to \infty$$

for some sequence $\{X_n\}_{n=1}^\infty \subseteq S_T$.

and that the Itô integral process $\{\int_0^t X dB\}_{t \in [0,T]}$ is well-defined as a limit in the sense of convergence in probability of $\int_0^t X_n dB$ as $n \to \infty$ for $t \in [0,T]$.

**Task 4.** Show that for a process $X \in P_T$ we have

$$\mathbf{P}\left\{ \int_0^T X(t)^2 \, dt = 0 \right\} = 1 \iff \mathbf{P}\left\{ \int_0^t X \, dB = 0 \right\} = 1 \quad \text{for} \quad t \in [0,T].$$

**Task 5.** Find stochastic processes $\{X(t)\}_{t \in [0,1]}$, $\{Y(t)\}_{t \in [0,1]}$ and $\{Z(t)\}_{t \in [0,1]}$ that belong to $E_1$, $P_1 \setminus E_1$ and $P_1^s$, respectively.

**Task 6.** Apply Itô's formula Theorem 4.17 in Klebaner's book to $f(X(t), Y(t))$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f(x, y) = g(x) y$ for a two times continuously differentiable function $g : \mathbb{R} \to \mathbb{R}$ and $X = Y = B$. Compare with what you get from applying the integration by parts formula Equation 4.57 in Klebaner’s book with $X = g(B)$ and $Y = B$. Derive from the comparison a new proof (without any explicit calculations other than applications of Itô's formula) of the property established in Example 4.23 in Klebaner's book that $[g(B), B](t) = \int_0^t g'(B(s)) \, ds$ for $t \geq 0$. 
Task 1

From the Cauchy Criterion we have $L^2$-convergence iff $\lim_{m,n \to \infty} E((X_m - X_n)^2) = 0$.

$\iff \lim_{m,n \to \infty} E(X_m X_n) = \xi$ exists $\Rightarrow$

$0 = \xi - 2\xi + \xi = \lim_{n \to \infty} E(X^2_m) - \lim_{n \to \infty} E(2X_m X_n) + \lim_{n \to \infty} E(X^2_n) = \lim_{n \to \infty} E((X_m - X_n)^2) \Rightarrow$

$L^2$-convergence.

$\Rightarrow L^2$-convergence $X_n \to_{L^2} X \Rightarrow \lim_{n \to \infty} E(X^2_n) = E(X^2)$ by the solved exercises to Chapter 4.
Putting this together with the Cauchy Criterion we get $E(X_m X_n) = \frac{1}{2}E(X^2_m) + \frac{1}{2} E(X^2_n) - \frac{1}{2} E(X^2_m - X_n)^2 \to \frac{1}{2} E(X^2) + \frac{1}{2} E(X^2) - 0$ as $m,n \to \infty$.

Task 2

For $x \in E_T$ we have $\int_0^T X_n dB - \int_0^T X dB$ as $n \to \infty$ for $\{X^2_n\}_{n=1}^{\infty} \subseteq S_T$ with $\int_0^T E((X_n(t) - X(t))^2) dt \to 0$.

By the isometry property for $X_n \in S_T$ together with the fact that $Y \to_{L^2} Y \Rightarrow E(Y^2_n) \to E(Y^2)$ (see above) it follows that $E((S_T^T X dB)^2)$

$= E((S_T^T X_n dB)^2) = S_T^T E((X_n(t))^2) = S_T^T E((X(t) - X(t))^2) dt + 2 S_T^T E((X_n(t) - X(t))^2) dt + S_T^T E((X(t))^2) dt \to S_T^T E((X(t))^2) dt$

since $|S_T^T E((X_n(t) - X(t))^2) dt| \leq S_T^T E((X_n(t) - X(t))^2) (X(t))^2) dt$.

$\leq S_T^T E((X(t))^2) dt + S_T^T E((X(t))^2) dt \to 0.$
Task 3

It is enough to show that for each \( n \in \mathbb{N} \) there exists \( X_n \in \mathcal{S}_T \) such that
\[
P( S_0^T (X_n(t) - X(t))^2 dt > \frac{r}{n} ) < \frac{1}{n}.
\]
No, we take \( Y_n \in \mathcal{E}_T \) such that
\[
P( S_0^T (Y_n(t) - X(t))^2 dt > \frac{r}{n} ) < \frac{1}{2n}
\]
and \( X_n \in \mathcal{S}_T \) such that
\[
E( S_0^T (X_n(t) - Y_n(t))^2 dt ) < \frac{r}{8n^2}.
\]
Then
\[
P( S_0^T (X_n(t) - X(t))^2 dt > \frac{r}{n} ) \\ + P( S_0^T (Y_n(t) - Y(t))^2 dt > \frac{r}{2n} ) \\ + \frac{1}{2n} < \frac{r}{8n^3} + \frac{1}{2n} = \frac{1}{n}.
\]

Task 4

\( \Rightarrow \) If \( P( S_0^T X(t)^2 dt = 0 ) = 1 \) for \( t \in [0, T] \), then by continuity of \( \{ S_0^T X(t)^2 dB^0 = \int_0^T \} \), we have
\[
P( S_0^T X dB = 0 + t \in [0, T] ) = 1,
\]
which implies that \( S_0^T X dB, S_0^T X dB = S_0^T X(t)^2 dt = 0 \) \( \text{wp} \ 1 \).

\( \Leftarrow \) If \( P( S_0^T X dB = 0 ) = 1 \) for \( t \in [0, T] \), then by continuity of \( \{ S_0^T X(t)^2 dB^0 = \int_0^T \} \), we have
\[
P( S_0^T X(t)^2 dt \geq 0 + t \in [0, T] ) = 1,
\]
which implies that \( S_0^T X dB, S_0^T X dB = S_0^T X(t)^2 dt = 0 \) \( \text{wp} \ 1 \).

Task 5

\( E_1 \) \( B \in \mathcal{E}_T \) since \( S_0^T (B(t)^2) dt = S_0^T + dt = \frac{1}{2} \).

\( P_{E_1} \) \( e^{B_2} \in P_{E_1} \) since the process is continuous with
\[
E( e^{B(t)^2} ) = E( e^{N(0,1)^2} ) = \int_{-\infty}^{\infty} e^{x^2} e^{-x^2/2t} dx = \infty \quad \text{for} \ t \in [\frac{1}{2}, 1] \text{making} \ S_0^T E( e^{B(t)^2} ) dt = \infty.
\]

\( P_1 \) Any non-adapted to \( \{ \mathcal{F}_t \}_{t \geq 0} \) works fine [such as, e.g., \( B(t) + \xi \) where \( \xi \) non-deterministic and independent of \( B \)] as well as, e.g., the (non-random) process \( X(t) = (t - \frac{1}{2})^2 \text{for} t \in [0, \frac{1}{2}] \), \( X(\frac{1}{2}) = 0 \), \( X(t) = (t - \frac{1}{2})^2 \text{for} t \in (\frac{1}{2}, 1] \).
By Ito's formula for \( f(X(t),Y(t)) \) with \( f(x,y) = g(x)y \) and \( X = Y = B \) we have:

\[
\begin{align*}
&\quad d \left( g(B(t))B(t) \right) = \\
&= g'(B(t))B(t)dB(t) + g(B(t))dB(t) + \frac{1}{2} g''(B(t))B(t)^2dt \\
&\quad + g'(B(t))dt \\
&\text{while integration by parts for} \quad g(B) \quad \text{and} \quad B \\
&\quad \text{gives} \quad d \left( g(B(t))B(t) \right) = dg(B(t))B(t) \\
&\quad + g(B(t))dB(t) + d \left[ g(B(t)), B(t) \right] = g'(B(t))B(t)dB(t) \\
&\quad + \frac{1}{2} g''(B(t))B(t)dt + g(B(t))dB(t) + d \left[ g(B(t)), B(t) \right].
\end{align*}
\]

Comparing these two results we conclude that

\[
d \left[ g(B(t)), B(t) \right] = g'(B(t))dt.
\]