Home Exercises for Chapter 5 in Klebaner's Book

Througout this set of exercises $B = \{B(t)\}_{t\geq 0}$ denotes Brownian motion.

Task 1. Find a diffusion type SDE that has a well-defined and unique strong solution, but that does not satisfy the conditions of Theorem 5.4 or Theorem 5.5 in Klebaner's book.

Task 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a two times continuously differentiable and strictly increasing function. Find a diffusion type SDE

$$dX(t) = \mu(X(t))\,dt + \sigma(X(t))\,dB(t) \quad \text{for } t > 0, \quad X(0) = f(0),$$

that has solution $X(t) = f(B(t))$.

Task 3. Show by means of direct calculation and/or inspection (not using Theorem 5.6 in Klebaner's book) that the solution given by Equation 5.13 in Klebaner's book to the Langevin equation 5.12 in Klebaner's book is a Markov process.

Task 4. Consider a stochastic processes $\{X(t)\}_{t\geq 0}$ that is adapted to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and satisfies

$$\mathbf{E}\{|X(t)|\} < \infty \quad \text{for } t \geq 0 \quad \text{and} \quad \mathbf{E}\{X(t)|X(s)\} = X(s) \quad \text{for } 0 \leq s \leq t.$$

Which is the most restrictive of the further requirements that X is a martingale and that X is a Markov process?

Task 5. The solution $\{X(t)\}_{t\geq 0}$ to the Langevin equation 5.12 in Klebaner's book is a Markov process with transition probability density function

$$p(t, x, y) = \frac{d}{dy}\mathbf{P}\{X(t+s) \leq y | X(s) = x\} = \frac{\sqrt{\alpha}}{\sqrt{\pi\,(1 - \mathrm{e}^{-2\alpha t})}\,\sigma}\exp\left\{-\frac{\alpha\,(y - x\,\mathrm{e}^{-\alpha t})^2}{\sigma^2\,(1 - \mathrm{e}^{-2\alpha t})}\right\}$$

for $s \geq 0$, $t > 0$ and $x, y \in \mathbb{R}$. (This can be verified, e.g., by means of use of Equation 5.13 in Klebaner's book.) Suppose that we know the value of the parameter $\alpha > 0$, but that we want to do a maximum likelihood estimation of the value of the parameter $\sigma > 0$ using observations $\{x_i\}_{i=0}^n$ of the process values $\{X(i)\}_{i=0}^n$. Find that maximum likelihood estimator.

Solutions to Home Exercises on Chapter 5

Task 1 The simplest examples are ODE like $dX(t) = -X(t)^2 dt$, $X(0) = 1$ with solution $X(t) = (1+t)^{-1}$. A truly SDE example would be $dX(t) = -X(t)^3 dt + dB(t)$.

Task 2 $X(t) = f(B(t)) \Rightarrow dX(t) = f'(B(t))dB(t) + \frac{1}{2}f''(B(t))dt$
$= \frac{1}{2}f''(f^{-1}(X(t)))dt + f'(f^{-1}(X(t)))dB(t)$ so $\mu(x) = \frac{1}{2}f''(f^{-1}(x))$, $\sigma(x) = f'(f^{-1}(x))$

Task 3 $X(t) = e^{-\alpha t}(x_0 + \int_0^t e^{\alpha r} dB(r)) = e^{-\alpha(t-s)} e^{-\alpha s}(x_0 + \int_0^s e^{\alpha r} dB(r))$
$+ e^{-\alpha t} \int_s^t e^{\alpha r} dB(r) = e^{-\alpha(t-s)} X(s) + e^{-\alpha t} \int_s^t e^{\alpha r} dB(r)$.

Task 4 $P(X(t) \in A | \mathcal{F}_s^X) = P(X(t) \in A | X(s)) \Rightarrow E(X(t) | \mathcal{F}_s^X) = E(X(t)|X(s))$
$= X(s)$ while $E(X(t)|\mathcal{F}_s^X) = X(s) = E(X(t)|X(s))$ does not imply Markov so martingale is the leas demanding additional assumption and Markov most restrictive.

Task 5 $f_{X(1),\ldots,X(n)}(x_1,\ldots,x_n) = \prod_{i=1}^{n} p(1, x_{i-1}, x_i)$

$$= \prod_{i=1}^{n} \frac{\sqrt{\alpha}}{\sqrt{\pi(1-e^{-2\alpha})}\,\sigma} \exp\left(-\frac{\alpha(x_i - e^{-\alpha}x_{i-1})^2}{\sigma^2(1-e^{-2\alpha})}\right)$$

$$= \frac{\alpha^{n/2}}{\pi^{n/2}(1-e^{-2\alpha})^{n/2}\sigma^n} \exp\left(-\frac{\alpha}{\sigma^2}\sum_{i=1}^{n}\frac{(x_i - e^{-\alpha}x_{i-1})^2}{(1-e^{-2\alpha})}\right)$$

with derivativer wrt σ given by

$$-\frac{n}{\sigma} + \frac{2\alpha}{\sigma^3}\sum_{i=1}^{n}\frac{(x_i - e^{-\alpha}x_{i-1})^2}{(1-e^{-2\alpha})}$$ which is zero for

$$\sigma = \hat{\sigma} = \sqrt{\frac{2\alpha}{n}\sum_{i=1}^{n}\frac{(x_i - e^{-\alpha}x_{i-1})^2}{(1-e^{-2\alpha})}}$$