Home Exercises for Chapter 6 in Klebaner's Book

Througout this set of exercises $B = \{B(t)\}_{t\geq 0}$ denotes Brownian motion.

Task 1. Consider the CKLS SDE from Exercise 4 of Exercise session 5

$$dX(t) = (\alpha + \beta X(t))\, dt + \sigma X(t)^{\gamma}\, dB(t) \quad \text{for } t > 0, \quad X(0) = X_0,$$

with parameters $\alpha, \sigma > 0$, $\beta \geq 0$ and $\gamma > 1$, and where X_0 has the stationary distribution (according to Exercise 4 of Exercise session 5) so that the solution $\{X(t)\}_{t\geq 0}$ is a stationary process (see Task 2 below). Show that

$$\mathbf{E}\left\{\int_0^t X(s)^{\gamma}\, dB(s)\right\} = C\,t \quad \text{for } t \geq 0,$$

for some strictly negative constant $C < 0$.

Task 2. Let $\{X(t)\}_{t\geq 0}$ be a time homogeneous diffusion process that has a stationary distribution and is started according to that stationary distribution at time $t = 0$. Prove that X is a stationary process, which is to say that

$$\mathbf{P}\{X(t_1+h) \leq x_1, \ldots, X(t_n+h) \leq x_n\} = \mathbf{P}\{X(t_1) \leq x_1, \ldots, X(t_n) \leq x_n\}$$

for $0 < t_1 < \ldots < t_n$ and $h > 0$.

Task 3. Find three SDE that explode, that display transcience but not explosion, and that display recurrence, respectively, but do not feature to exemplify these properties in Klebaner's book. Also, find three SDE where the issue whether the above three mentioned properties hold depends on the starting value of the SDE.

Task 4. Find a PDE that is solved by the fair price $p(x,t) = \mathbf{E}\{\max\{X(T) - K, 0\} | X(t) = x\}$ (for a constant $K > 0$) of an European call option at time $t \in [0, T)$ for an asset price $\{X(t)\}_{t\in[0,T]}$ given by the CKLS SDE from Exercise 4 of Exercise session 5.

Solutions to Home Exercises on Chapter 6

Task 1 $\varphi = E(X(t)) = E(X(0)) + \int_0^t (\alpha + \beta E(X(s)))\,ds + E\left(\int_0^t \sigma X^\gamma dB\right)$

$\Rightarrow E\left(\int_0^t \sigma X^\gamma dB\right) + \int_0^t (\alpha + \beta\varphi)\,ds = 0$ where $C = -(\alpha + \beta\varphi)$ is negative because of assumptions made in task.

Task 4 $p(x,t) = E(g(X(T)) \mid X(t) = x)$ with $g(x) = \max\{x - K, 0\}$ and $X(t)$ solving SDE with $\mu(x,t) = \alpha + \beta x$, $\sigma(x,t) = \sigma x^\gamma$.
This means that $p(x,t)$ solves PDE

$$L_t p(x,t) + \frac{\partial p(x,t)}{\partial t} = 0 \text{ for } t \in [0,T],\ p(x,T) = \max\{x-K,0\}$$

Task 2 $f_{X(t_1+h),\ldots,X(t_n+h)}(x_1,\ldots,x_n) = \int_{-\infty}^{+\infty} \prod_{i=1}^n p(t_i - t_{i-1}, x_i - x_{i-1})\,\pi(x_0)\,dx_0$
where the right-hand side does not depend on h (and $t_0 = 0$).

Task 3 $dX(t) = \alpha X(t)dt + \sigma\,dB(t)$ with $\alpha > 0$ is transient according to
easy check of criteria but does not explode as we can solve it as an Langevin equation.

$dX(t) = -\text{sign}(X(t))X(t)^2 + dB(t)$ is reccurent as $\exp\left(-\int_0^x \frac{2\mu(y)}{\sigma(y)^2}\right) = \exp\left(\frac{2\,\text{sign}(x)\,x^3}{3}\right)$ making $\int_0^\infty$ and $\int_{-\infty}^0$ of it both infinite.

$dX(t) = -X(t)^2 + dB(t)$ explodes as $\int_x^0 \frac{1}{\sigma(y)^2}\left(\int_0^y \frac{2\mu(z)}{\sigma(z)^2}dz\right)dy$
$= \int_x^0 e^{-2y^3/3}dy$ and $\int_{-\infty}^0 e^{2x^3/3}\int_x^0 e^{-2y^3/3}dy\,dx < \infty$

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In[4]:= Integrate[
   Exp[2 * x^3 / 3] * Integrate[Exp[-2 * y^3 / 3], {y, x, 0}], {x, -Infinity, 0}]
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$$\text{Out[4]=}\ \frac{\left(-\frac{1}{2}\right)^{1/3}\sqrt{\pi}\ \text{Gamma}\left[\frac{1}{6}\right]}{3\ \ 3^{5/6}} - \left(-\frac{3}{2}\right)^{2/3}\text{Gamma}\left[\frac{4}{3}\right]^2$$

To have these properties depend on starting value select SDE of one of these types in one region and another type in another and make it impossible for SDE to travel between regions.