N.1

Home Exercises on Numerical Methods

Througout this set of exercises $B=\{B(t)\}_{t\geq 0}$ denotes Brownian motion.

Task 1. It is not hard at all to find SDE for which the Euler (Euler-Maruyama) method collapses with a very high probability – one exemple would be the CKLS SDE (see Task 2 below) with a large γ coefficient. For such SDE it is usually the case that a so called implicit method works much better: With the notation of Section 1.2 in Stig Larsson's lecture notes, the so called fully implicit Euler method is given by $Y_0=X_0$ and

$$Y_{n+1}=Y_n+\Big(\mu(Y_{n+1},t_{n+1})-\Big(\sigma\frac{\partial\sigma}{\partial x}\Big)(Y_{n+1},t_{n+1})\Big)\int_{t_n}^{t_{n+1}}ds+\sigma(Y_{n+1},t_{n+1})\int_{t_n}^{t_{n+1}}dB\,.$$

for $n\geq 0$. Explain why this method does not take the simpler form

$$Y_0=X_0\quad\text{and}\quad Y_{n+1}=Y_n+\mu(Y_{n+1},t_{n+1})\int_{t_n}^{t_{n+1}}ds+\sigma(Y_{n+1},t_{n+1})\int_{t_n}^{t_{n+1}}dB\quad\text{for } n\geq 0.$$

Task 2. Demonstrate by means of simulations the fact that for the CKLS SDE

$$dX(t)=(\alpha+\beta\,X(t))\,dt+\sigma\,X(t)^\gamma\,dB(t)\quad\text{for } t>0,\quad X(0)=x_0,$$

with some choice of parameters $\alpha,\beta,\sigma,x_0>0$ and a large $\gamma>1$ (e.g., $\gamma=5$), the Euler method breaks down while the fully implicit Euler method works.

Task 3. Show by means of simulations of the SDE in Exercise 6 of Exercise session 4

$$dX(t)=\Big(\sqrt{1+X(t)^2}+\frac{X(t)}{2}\Big)\,dt+\sqrt{1+X(t)^2}\,dB(t)\quad\text{for } t>0,\quad X(0)=0,$$

that the Milstein method converges quicker than the Euler method. Are the conditions for convergence of the Euler method satisfied?

Task 4. Given constants $\mu\in\mathbb{R}$ and $\sigma>0$, find the solution $f(x,t)$ to the PDE

$$\mu\,\frac{\partial f(x,t)}{\partial x}+\frac{\sigma^2}{2}\,\frac{\partial^2 f(x,t)}{\partial x^2}+\frac{\partial f(x,t)}{\partial t}=0\quad\text{for } t\in[0,T),\quad f(x,T)=x^2.$$

Task 5. Show that a solution $f(x,t)$ to the PDE

$$\mu(x,t)\,\frac{\partial f}{\partial x}+\frac{\sigma(x,t)^2}{2}\,\frac{\partial^2 f}{\partial x^2}+\gamma(x,t)\,f(x,t)+\frac{\partial f}{\partial t}=k(x,t)\quad\text{for } t\in[0,T),\quad f(x,T)=g(x),$$

must take the form (with obvious notation)

$$f(x,t)=\mathbf{E}\Big\{g(X(T))\,e^{\int_t^T\gamma(X(s),s)\,ds}-\int_t^T k(X(s),s)\,e^{\int_t^s\gamma(X(r),r)\,dr}ds\;\Big|\;X(t)=x\Big\}.$$

Solutions to Home Exercises on Numerical Methods

Task 1 The reason is that

$$\sum_{i=1}^{n}\left(\sigma(X(t_i),t_i)-\sigma(X(t_{i-1}),t_{i-1})\right)\left(B(t_i)-B(t_{i-1})\right) \to [\sigma(X(t),t),B(t)]$$

$$=\int_0^t d\sigma(X(s),s)\,dB(s) = \int_0^t\left(\sigma_x'\,dX(s)+\tfrac{1}{2}\sigma_{xx}''\,d[X,X](s)+\sigma_t'\,ds\right)dB(s)$$

$$=\int_0^t(\sigma_x'\sigma)(X(s),s)\,ds$$

Task 2

Numerical Home Exercises Task 2

dX (t) = (1 + X (t)) dt + X (t) ^5 dB (t), X (0) = 1

Attempt to simulate CKLS SDE with explicit Euler Maruyama scheme and h = 0.001

```
In[59]:= h = 0.001; RepsToFailure = {}; Attempts = 10;

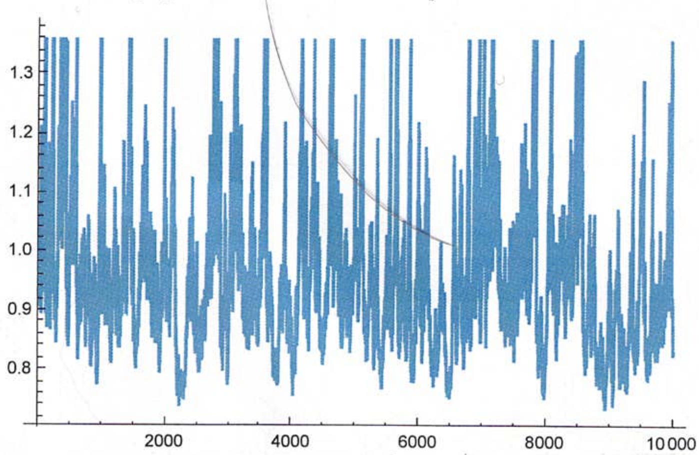
In[60]:= For[i = 1, i ≤ Attempts, i++, X = {1};
  step = 1;
  While[Abs[X[[step]]] ≤ 1 / h, AppendTo[X, X[[step]] + (1 + X[[step]]) * h +
      X[[step]] ^5 * Random[NormalDistribution[0, Sqrt[h]]]];
   step = step + 1];
  AppendTo[RepsToFailure, step]];
RepsToFailure

Out[60]= {759, 656, 233, 998, 740, 194, 611, 513, 365, 1006}
```

In[62]:= Simulation of CKLS SDE with imlicit Euler Maruyama scheme for 0 < t < 10 with h = 0.001

```
In[70]:= For[step = 1;
  X = {1}, step ≤ 10 000, step++, dB = Random[NormalDistribution[0, Sqrt[h]]];
  AppendTo[X,
   x /. NSolve[x == X[[step]] + (1 + x - 5 * x^9) * h + x^5 * dB, x, Reals][[1]]]];
ListPlot[X, PlotJoined → True]
```

Out[70]=



Task3

Numerical Home Exercise Task 3

```
dX (t) = (Sqrt[1 + X (t) ^2] + X (t) / 2) dt + Sqrt[1 + X (t) ^2] dB (t), X (0) = 0
```

Manifacturing of BM

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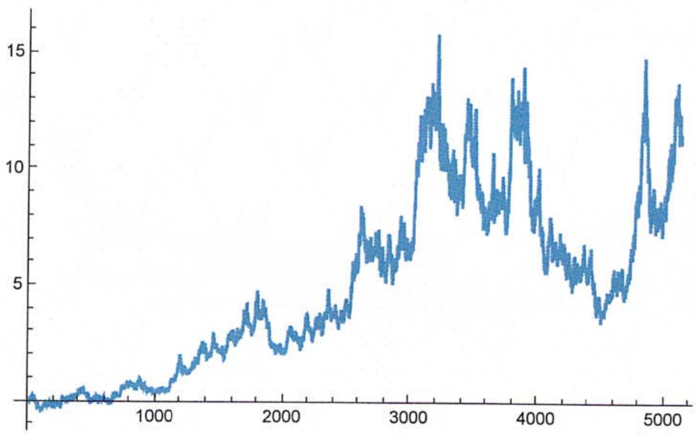
In[48]:= Steps = 2^10;
h = N[1 / Steps];
dB = Table[Random[NormalDistribution[0, Sqrt[h]]], {5 * Steps}];
```

Plot of exact solution X (t) = Sinh[B (t) + t] for 0 < t < 5

```
In[49]:= For[i = 1;
  X = {0};
  B = {0}, i ≤ 5 * Steps, i++, AppendTo[B, B[[i]] + dB[[i]]];
  AppendTo[X, Sinh[B[[i + 1]] + i * h]]];
ListPlot[X, PlotJoined → True]
```

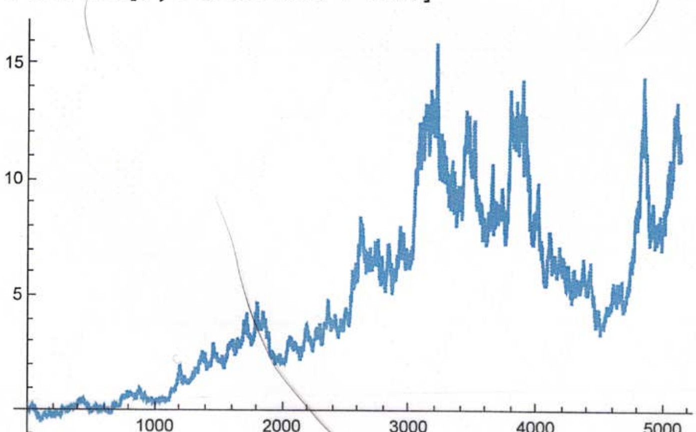
Out[49]=



Plot of numerical solution with Euler method for 0 < t < 5 with h = 1 / 2^10

```
In[50]:= For[i = 1;
  Y = {0}, i ≤ 5 * Steps, i++, AppendTo[Y,
   Y[[i]] + (Sqrt[1 + Y[[i]]^2] + Y[[i]] / 2) * h + Sqrt[1 + Y[[i]]^2] * dB[[i]]]];
ListPlot[Y, PlotJoined → True]
```

Out[50]=



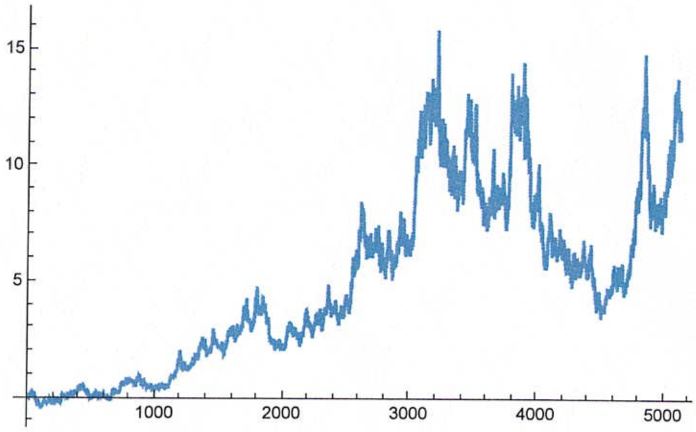
The conditions for convergence of Euler method global Lipschitz coefficients and global linear growth are clearly satisfied.

Task 3 continued

Plot of numerical solution with Milstein method for 0 < t < 5 with h = 1 / 2^10

```
In[52]:= For[i = 1;
  Z = {0}, i ≤ 5 * Steps, i++, AppendTo[Z, Z[[i]] + (Sqrt[1 + Z[[i]]^2] + Z[[i]] / 2) * h +
     Sqrt[1 + Z[[i]]^2] * dB[[i]] + Z[[i]] * (dB[[i]]^2 - h) / 2]];
 ListPlot[X, PlotJoined → True]
```

Out[52]=



Max[Abs[exact - Euler] , Abs[exact - Milstein]] for h = 1 / 2^n and 5 ≤ n ≤ 16

```
In[58]:= For[n = 5;
  Diff = {}, n ≤ 16, n++, Steps = 2^n;
  h = N[1 / Steps];
  dB = Table[Random[NormalDistribution[0, Sqrt[h]]], {5 * Steps}];
  For[i = 1;
   X = {0};
   B = {0}, i ≤ 5 * Steps, i++, AppendTo[B, B[[i]] + dB[[i]]];
   AppendTo[X, Sinh[B[[i + 1]] + i * h]]];
  For[i = 1;
   Y = {0}, i ≤ 5 * Steps, i++, AppendTo[Y,
    Y[[i]] + (Sqrt[1 + Y[[i]]^2] + Y[[i]] / 2) * h + Sqrt[1 + Y[[i]]^2] * dB[[i]]]];
  For[i = 1;
   Z = {0}, i ≤ 5 * Steps, i++, AppendTo[Z, Z[[i]] + (Sqrt[1 + Z[[i]]^2] + Z[[i]] / 2) *
      h + Sqrt[1 + Z[[i]]^2] * dB[[i]] + Z[[i]] * (dB[[i]]^2 - h) / 2]];
  AppendTo[Diff, {Max[Abs[X - Y]], Max[Abs[X - Z]]}]];
 Diff

Out[58]= {{17.5903, 2.14912}, {40.469, 3.69162}, {7.20581, 2.40503},
  {0.122901, 0.036252}, {3.44348, 0.0832462}, {5.94456, 0.168757},
  {0.0315548, 0.00425817}, {0.214378, 0.0146471}, {0.316021, 0.00423048},
  {2.91505, 0.0146062}, {0.164052, 0.0033169}, {1.75225, 0.00587775}}
```

Task 4

$\mu(x,t) = \mu$, $\sigma(x,t) = \sigma$ and $g(x) = x^2$; Theorem 6.6 gives

$f(x,t) = E(X(T)^2 / X(t) = x)$ with $dX(t) = \mu\,dt + \sigma\,dB(t)$

so $f(x,t) = E((\sigma B(T) + \mu T)^2 \mid \sigma B(t) + \mu t = x) = E((\sigma(B(T) - B(t)) + \mu(T-t))^2)$

$+ 2E((\sigma B(T-t) + \mu(T-t))x) + x^2 = \sigma^2(T-t) + \mu^2(T-t)^2 + 2\mu(T-t)x + x^2$

$\boxed{\text{Task5}}$ $d\left(f(X(s),s)\exp\left(\int_t^s \gamma(X(r),r)\,dr\right)\right)$

$= \left((\partial_s f)ds + (\partial_x f)dX + \frac{1}{2}(\partial^2_{xx} f)d[X,X](t) + f\gamma ds\right)\exp$

$= \left((\partial_s f)ds + (\partial_x f)(\mu ds + \sigma dB) + \frac{1}{2}(\partial^2_{xx} f)\sigma^2 ds + f\gamma ds\right)\exp$

$= \left((\partial_x f)\sigma dB + k ds\right)\exp$

so that

$$f(X(T),T)\exp\left(\int_t^T \gamma(X(r),r)dr\right) - f(X(t),t)\exp\left(\int_t^t \gamma(X(r),r)\,dr\right)$$

$$= g(X(T))\exp\left(\int_t^T \gamma(X(r),r)dr\right) - f(X(t),t)$$

equals

$$\int_t^T (\partial_x f)(X(s),s)\,\sigma(X(s),s)\exp\left(\int_t^s \gamma(X(r),r)dr\right)dB(s)$$

$$+ \int_t^T k(X(s),s)\exp\left(\int_t^s \gamma(X(r),r)\,dr\right)ds$$

Upon rearranging and taking conditional expectation with respect to $X(t)=x$ it follows that

$$f(x,t) = \text{the claim} + E\left(\int_t^T \partial_x f(X(s),s))\sigma(X(s),s)\exp\left(\int_t^s \gamma(X(r),r)dr\right)dB(s) \Big/ X(t)=x\right)$$

which equals the claim by zero-mean property of Itô integrals (together with Markov property of X).