1 Preliminaries from Calculus

1.1 Variation of a Function

Definition 1.1. The variation $V_g([a,b])$ of a function $g:[a,b]\to\mathbb{R}$ over the interval $[a,b]$ is defined as

$$V_g([a,b]) = \lim\Big\{\sum_{i=1}^{n}|g(t_i^n)-g(t_{i-1}^n)| : a=t_0^n<t_1^n<\ldots<t_n^n=b,\ n\in\mathbb{N},\ \max_{1\le i\le n} t_i^n-t_{i-1}^n\downarrow 0\Big\}$$

whenever the limit[1] exists.

When $V_g([a,b])<\infty$ we say that g has finite variation (FV).

Functions and random processes in stochastic calculus are always either (at least) continuous from the right with existing left limits (cádlág[2]) or continuous from the left with existing right limits (cáglád) and therefore have a well-defined variation[3].

For a function $g:[0,\infty)\to\mathbb{R}$ we use the short-hand notation $V_g(t)$ for $V_g([0,t])$.

Theorem 1.2. 1. $V_g([a,c]) = V_g([a,b]) + V_g([b,c])$ for $a\le b\le c$,

2. $V_g([a,t])$ is an increasing function of $t\ge a$,
3. $V_g([a,b])\le V_g([c,d])$ for $[a,b]\subseteq[c,d]$,
4. $V_g([a,b])\ge|g(a)-g(b)|$ with equality if and only if g is monotone,
5. $V_g(t)=\sum_{0\le s\le t}|\Delta g(s)|$ for a pure jump function $g(t)=\sum_{0\le s\le t}\Delta g(s)$,
6. $V_g([a,b])=\int_a^b|g'(t)|\,dt$ for $g:[a,b]\to\mathbb{R}$ continuously differentiable,
7. g is FV if and only a difference between two increasing functions.

A continuous function need not be FV: For example, the stochastic process Brownian motion we Chapter 3 is continuous but has infinite variation over each interval.

Proof. 1. Follows noting that it is no restriction to assume that b is a member of the meshes $a=t_0^n<t_1^n<\ldots<t_n^n=c$ used to compute $V_g([a,c])$.

2-5. 2-3 follow from 1 while 4-5 are obvious from the definition.

6. By the mean value theorem together with Riemann sum approximation we have

[1] If you wonder a little about the limit you are right: This actually is so called net convergence.

[2] Continuite á droit, limite á gauche.

[3] Prove cádlág implies uniform right continuity and use this to show the limit equals sup of the same thing.

$$V_g([a,b]) = \lim \sum_{i=1}^{n} |g(t_i^n) - g(t_{i-1}^n)| = \lim \sum_{i=1}^{n} |g'(s_i^n)|\,(t_i^n - t_{i-1}^n) = \int_a^b |g'(t)|\,dt$$

for some $s_i^n \in [t_{i-1}^n, t_i^n]$, with the limit as $a = t_0^n < t_1^n < \ldots < t_n^n = b$ and $\max_{1\le i\le n} t_i^n - t_{i-1}^n \downarrow 0$.

7. If g is the difference between two increasing function $g(t) = i_1(t) - i_2(t)$ we have

$$\sum_{i=1}^{n} |g(t_i^n) - g(t_{i-1}^n)| \le \sum_{i=1}^{n} \big((i_1(t_i^n) - i_1(t_{i-1}^n)) + (i_2(t_i^n) - i_2(t_{i-1}^n))\big) = (i_1(b) - i_1(a)) + (i_2(b) - i_2(a)).$$

If on the other hand $V_g([a,b]) < \infty$, then we have $g(t) = V_g([a,t]) - (V_g([a,t]) - g(t))$ where $V_g([a,t])$ is increasing by 2 and where by 4, for $a \le s \le t$,

$$(V_g([a,t]) - g(t)) - (V_g([a,s]) - g(s)) = V_g([s,t]) - (g(t) - g(s)) \ge V_g([s,t]) - |g(t) - g(s)|. \quad \square$$

1.2 Quadratic Variation of a Function

Definition 1.3. The quadratic covariation or simply covariation $[f,g]([a,b])$ between two functions $f,g : [a,b] \to \mathbb{R}$ over the interval $[a,b]$ is defined as

$$\lim \Big\{ \sum_{i=1}^{n} (f(t_i^n) - f(t_{i-1}^n))\,(g(t_i^n) - g(t_{i-1}^n)) : a = t_0^n < t_1^n < \ldots < t_n^n = b,\ n \in \mathbb{N},\ \max_{1\le i\le n} t_i^n - t_{i-1}^n \downarrow 0 \Big\}$$

whenever the limit exists.

The quadratic variation of $g : [a,b] \to \mathbb{R}$ is $[g]([a,b]) = [g,g]([a,b])$.

Functions and random processes in stochastic calculus always have a well-defined covariation[4].

We use the short-hand notation $[f,g](t)$ for $[f,g]([0,t])$ and $[g](t)$ for $[g]([0,t])$.

Theorem 1.4. If one of $f,g : [a,b] \to \mathbb{R}$ is continuous and the other is FV then $[f,g]([a,b]) = 0$.

In particular $[g]([a,b]) = 0$ if $g : [a,b] \to \mathbb{R}$ is continuous and FV.

Proof. For f continuous and g FV the uniform continuity of f gives

$$\Big| \sum_{i=1}^{n} (f(t_i^n) - f(t_{i-1}^n))\,(g(t_i^n) - g(t_{i-1}^n)) \Big| \le \max_{1\le i\le n} |f(t_i^n) - f(t_{i-1}^n)| \sum_{i=1}^{n} |g(t_i^n) - g(t_{i-1}^n)| \to 0 \cdot V_g([a,b]).$$

$\square$

Covariation is an inner product and a positive semi-definite symmetric bilinear form. Therefore it has the same properties as covariance:

[4] The proof involves advanced martingale theory and (although big words :)) cannot be made at home.

Theorem 1.5. 1. $[g]([a,b]) \geq 0$ (positivity),

2. $[f,g]([a,b]) = [g,f]([a,b])$ (symmetry),

3. $[\alpha\, f_1 + \beta\, f_2, g]([a,b]) = \alpha\, [f_1,g]([a,b]) + \beta\, [f_2,g]([a,b])$ (linearity),

4. $[f,g]([a,b]) = \frac{1}{2}\big([f{+}g]([a,b]){-}[f]([a,b]){-}[g]([a,b])\big)$ (polarization),

5. $[f,g]([a,b]) = \frac{1}{4}\big([f+g]([a,b]) - [f-g]([a,b])\big)$ (polarization),

6. $|[f,g]([a,b])| \leq \sqrt{[f]([a,b])\,[g]([a,b])}$ (Cauchy-Schwarz inequality).

Proof. 1-5. By inspection of the definitions.

6. Proved in the same way as the corresponding statement for covariances and variances. □

1.3 Riemann-Stieltjes Integral

Definition 1.6. The Riemann-Stieltjes (RS) integral $\int_a^b f(t)\,dg(t)$ of a continuous function $f : [a,b] \to \mathbb{R}$ wrt. an FV function $g : [a,b] \to \mathbb{R}$ is defined as

$$\lim\Big\{\sum_{i=1}^{n} f(s_i^n)\,(g(t_i^n){-}g(t_{i-1}^n)) : a{=}t_0^n{<}t_1^n{<}\ldots{<}t_n^n{=}b,\, s_i^n\in[t_{i-1}^n,t_i^n],\, n\in\mathbb{N},\, \max_{1\leq i\leq n} t_i^n{-}t_{i-1}^n{\downarrow}0\Big\}.$$

Under the stated conditions on f and g the above limit exists[5].

The ordinary Riemann integral is the special case of the RS integral with $g(x) = x$. This can be viewed as the Euclidian case while more general (than linear) choices of g corresponds to a non-Euclidian measure of (possibly signed) length $g(t) - g(s)$ of intervals $[s,t] \subseteq [a,b]$.

Theorem 1.7. 1. For both f and g continuous and FV we have

$$\int_a^b f\,dg + \int_a^b g\,df = f(b)g(b){-}f(a)g(a) \qquad \text{(integration by parts)}.$$

2. For f continuous and g continuously differentiable we have

$$\int_a^b f\,dg = \int_a^b f(t)\,\tfrac{dg}{dt}\,dt = \int_a^b f(t)\,g'(t)\,dt.$$

3. For f continuously differentiable and g continuous and FV we have

$$\int_a^b f'(g(t))\,dg(t) = \int_{g(a)}^{g(b)} f'(s)\,ds = f(g(b)){-}f(g(a)) \qquad \text{(change of variable)}.$$

Proof. 1. $\int_a^b f\,dg + \int_a^b g\,df \leftarrow \sum_{i=1}^{n} f(t_{i-1}^n)\,(g(t_i^n) - g(t_{i-1}^n)) + \sum_{i=1}^{n} g(t_i^n)\,(f(t_i^n) - f(t_{i-1}^n))$

[5] Argue as in the proof of Theorem 1.4 to see that the limsup and liminf are equal and most be finite.

$$= \sum_{i=1}^{n} (f(t_i^n)g(t_i^n) - f(t_{i-1}^n)g(t_{i-1}^n)) = f(b)g(b) - f(a)g(a).$$

2. By the mean value theorem and Riemann sum approximation as in 6 of Theorem 1.2.

3. For g continuously differentiable this follows from 2 and elementary inner derivative formula. By approximation with continuously differentiable functions (e.g., by convolution smoothing) this carries over to a more general continuous FV g. □

The RS integral also includes the concept of sum: To see this note that

$$\sum_{i=1}^{n} a_i = \int_{1/2}^{n+1/2} f\, dg$$

for any continuous f with $f(i) = a_i$ and $g(t) = \lfloor t \rfloor = i-1$ for $t \in [i-1, i)$, for $i = 1, \ldots, n+1$.

RS integrals are extended to infinite intervals as this is done for Riemann integrals.