3 Basic Stochastic Processes

3.1 Martingales

Definition 3.1. A stochastic process with parameter set T is a family $X = \{X(t)\}_{t\in T} = \{X(\omega,t)\}_{t\in T}$ of random variables on a probability space $(\Omega, \mathcal{F}, P)$.

A sample path of X is the function $X(\omega_0,\cdot) : T \to \mathbb{R}$ for a fixed $\omega_0 \in \Omega$.

In stochastic calculus we consider processes with (time) parameter set $[0,\infty)$ or $[0,T]$. We will present some concepts for the parameter set $[0,\infty)$ below but they are rephrased for parameter set $[0,T]$ by obvious modifications.

Definition 3.2.

1. A filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is an increasing family of σ-fields all contained in $\mathcal{F}$. When $(\Omega, \mathcal{F}, \mathbf{P})$ is equipped with a filtration we call $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ a filtered probability space.
2. A stochastic process $\{X(t)\}_{t\geq 0}$ is called adapted to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ if $X(t)$ is $\mathcal{F}_t$-measurable for $t \geq 0$.
3. The smallest filtration that $X = \{X(t)\}_{t\geq 0}$ is adapted to is the filtration generated by X itself $\{\mathcal{F}_t^X\}_{t\geq 0}$ given by $\mathcal{F}_t^X = \bigvee_{s\in[0,t]} \sigma(X(s))$.

Definition 3.3. Let $\{X(t)\}_{t\geq 0}$ be a stochastic process adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ such that $\mathbf{E}\{|X(t)|\} < \infty$ for $t \geq 0$. We say that

1. X is a martingale wrt. $\{\mathcal{F}_t\}_{t\geq 0}$ if $\mathbf{E}\{X(t)|\mathcal{F}_s\} = X(s)$ for $s \leq t$,
2. X is a submartingale wrt. $\{\mathcal{F}_t\}_{t\geq 0}$ if $\mathbf{E}\{X(t)|\mathcal{F}_s\} \geq X(s)$ for $s \leq t$,
3. X is a supermartingale wrt. $\{\mathcal{F}_t\}_{t\geq 0}$ if $\mathbf{E}\{X(t)|\mathcal{F}_s\} \leq X(s)$ for $s \leq t$.

Theorem 3.4.

1. (Doob-Lévy martingale) If Y is a random variable with $\mathbf{E}\{|Y|\} < \infty$ then $X(t) = \mathbf{E}\{Y|\mathcal{F}_t\}$ for $t \geq 0$ is a martingale.
2. Martingales have constant expectation while submartingales (supermartingales) have increasing (decreasing) expectations.
3. For X a martingale and g a convex function $g(X)$ is a submartingale.
4. A martingale X is a martingale wrt. $\{\mathcal{F}_t^X\}_{t\geq 0}$.

Proof. 1. By towering $\mathbf{E}\{X(t)|\mathcal{F}_s\} = \mathbf{E}\{\mathbf{E}\{Y|\mathcal{F}_t\}|\mathcal{F}_s\} = \mathbf{E}\{Y|\mathcal{F}_s\} = X(s)$.

2. For a submartingale (supermartingale) positivity and 10 in Theorem 2.14 give $\mathbf{E}\{X(t)\} = \mathbf{E}\{\mathbf{E}\{X(t)|\mathcal{F}_s\}\} \geq (\leq)\ \mathbf{E}\{X(s)\}$.

3. By Jensen's inequality $\mathbf{E}\{g(X(t))|\mathcal{F}_s\} \geq g(\mathbf{E}\{X(t)|\mathcal{F}_s\}) = g(X(s))$.

4. By towering $\mathbf{E}\{X(t)|\mathcal{F}_s^X\} = \mathbf{E}\{\mathbf{E}\{X(t)|\mathcal{F}_s\}|\mathcal{F}_s^X\} = \mathbf{E}\{X(s)|\mathcal{F}_s^X\} = X(s)$ as $\mathcal{F}_s^X \subseteq \mathcal{F}_s$. $\square$

3.2 Markov Processes

Definition 3.5. A stochastic process $\{X(t)\}_{t\geq 0}$ is a Markov process if

$$\mathbf{P}\{X(t) \in A|\mathcal{F}_s^X\} = \mathbf{P}\{X(t) \in A|\sigma(X(s))\} \quad \text{for } 0 \leq s \leq t \text{ and } A \in \mathcal{B}.$$

Theorem 3.6. Independent increment processes are Markov processes.

Proof. By 14 in Theorem 2.16

$$\begin{aligned}\mathbf{P}\{X(t) \in A|\mathcal{F}_s^X\} &= \mathbf{E}\{\mathbf{1}_{A-X(s)}(X(t)-X(s))|\mathcal{F}_s^X\} \\ &= \mathbf{E}\{\mathbf{1}_{A-X(s)}(X(t)-X(s))|\sigma(X(s))\} = \mathbf{P}\{X(t) \in A|\sigma(X(s))\}.\end{aligned}$$ $\square$

Definition 3.7. The transition CDF for a Markov process X is given by

$$\underline{P(y,t,x,s)} = \mathbf{P}\{X(t) \leq y|X(s) = x\} \quad \text{for } 0 \leq s \leq t \text{ and } x, y \in \mathbb{R}.$$

If an accompaning transition PDF $f_{X(t)|X(s)}(y|x)$ exists it is denoted $\underline{p(y,t,x,s)}$.

Definition 3.8. A Markov processes is time homogeneous if $\underline{P(t,x,y)} = P(y,t+s,x,s)$ and $\underline{p(t,x,y)} = p(y,t+s,x,s)$ do not depend on s.

For a time homogeneous Markov process $\{X(t)\}_{t\geq 0}$ with transition PDF $p(t,x,y)$ the joint PDF of $(X(t_1),\ldots,X(t_n))$ for $0 \leq t_1 < \cdots < t_n$ is given by

$$f_{X(t_1),\ldots,X(t_n)}(x_1,\ldots,x_n) = f_{X(t_1)}(x_1)\prod_{i=2}^{n} p(t_i - t_{i-1}, x_{i-1}, x_i) \quad \text{for } x_1,\ldots,x_n \in \mathbb{R}$$

if $X(t_1)$ has PDF $f_{X(t_1)}$. This is proved for BM (see below) in the exercises but the proof in general is by analogy. See also applied lecture. For $X(t_1) = x_1$ non-random we have

$$f_{X(t_2),\ldots,X(t_n)}(x_2,\ldots,x_n) = \prod_{i=2}^{n} p(t_i - t_{i-1}, x_{i-1}, x_i) \quad \text{for } x_2,\ldots,x_n \in \mathbb{R}.$$

3.3 Gaussian Processes

Definition 3.9. A random process $\{X(t)\}_{t\in T}$ is Gaussian (or normal) if, for $a_1,\ldots,a_n\in\mathbb{R}$, $t_1,\ldots,t_n\in T$ and $n\in\mathbb{N}$, $\sum_{i=1}^n a_i X(t_i)$ is normal distributed.

A Gaussian process has a normal distributed process value $X(t)$ at each time $t\in T$. But the requirement that a process is Gaussian is a much more demanding requirement than just individual process values being Gaussian: See the exercises for an example.

Here are some important but not very hard to prove facts about Gaussian processes[1]:

- The probability distribution for any vector of $(X(t_1),\ldots,X(t_n))$ of Gaussian process values is completely determined by the mean function $T\ni t\curvearrowright \mathbf{E}\{X(t)\}\in\mathbb{R}$ and the covariance function $T\times T\ni(s,t)\curvearrowright \mathbf{Cov}\{X(s),X(t)\}\in\mathbb{R}$ of the process.
- A collection of Gaussian process values with mutual covariances zero are independent.
- A Gaussian process is weak (wide) sense stationary if and only if (strictly) stationary.

3.4 Brownian Motion

Definition 3.10. Brownian motion (BM) (also called Wiener process) is a stochastic process $\{B(t)\}_{t\geq 0}$ with the following properties:

- $B(0)=0$,
- $B(t+s)-B(s)$ is independent of $\mathcal{F}_s^B$ for $s,t\geq 0$ (independent increments),
- $B(t+s)-B(s)$ is $\mathrm{N}(0,t)$-distributed for $s,t\geq 0$ (stationary increments),
- $\{B(t)\}_{t\geq 0}$ have continuous sample paths (continuity).

BM can also be started at $B(0)=x$ with the other three axioms kept and one writes $\{B^x(t)\}_{t\geq 0}$ for BM started at x. Alternatively one can write $\mathbf{P}_x\{B(t)\leq y\}$ for $\mathbf{P}\{B^x(t)\leq y\}$. Probabilities for $\{B^x(t)\}_{t\geq 0}$ coincide with those of $\{B^0(t)+x\}_{t\geq 0}$ – space homogeneity.

There is a tradition to be unspecific about the starting point for BM as well as more general time homogeneous Markov processes. The reason is that the transition probabilities have a live of their own making it meaningful to talk about, for example,

$$P(y,t,x,s)=P(y,t-s,x,0)=P(t-s,x,y)$$

[1]They belong to a basic stochastic processe course rather than stochastic calculus so are not proved here.

even if $X(0) \neq x$. This can be less confusing if one is unspecific about the starting point.

If in an BM calculation the starting point is needed but not given it is assumed 0.

Example 3.1. For independent increment processes as BM calculations are often done by expressing things as sums of independent increments: E.g.,

$$\begin{aligned}\mathbf{P}\{B(1) \leq x,\, B(2) \leq y,\, B(3) \leq z\} &= \mathbf{P}\{\xi_1 \leq x,\, \xi_1+\xi_2 \leq y,\, \xi_1+\xi_2+\xi_3 \leq z\} \\ &= \int_{u=-\infty}^{u=x} \phi(u) \int_{v=-\infty}^{v=y-u} \phi(v) \int_{w=-\infty}^{w=z-u-v} \phi(w)\, dwdvdu\end{aligned}$$

for $x, y, z \in \mathbb{R}$, where ξ_1, ξ_2 and ξ_3 are independent standard normal.

In a similar fashion it is proven that BM is Gaussian:

Theorem 3.11. BM is zero-mean Gaussian with $\mathbf{Cov}\{B(s), B(t)\} = s \wedge t$.

Proof. $B(t) = (B(t) - B(0)) + B(0) = B(t) - B(0)$ is zero-mean $\mathrm{N}(0, t)$ with

$$\mathbf{Cov}\{B(s), B(t)\} = \mathbf{Cov}\{B(s), B(t) - B(s)\} + \mathbf{Cov}\{B(s), B(s)\} = 0 + \mathbf{Var}\{B(s)\} = s$$

for $0 \leq s \leq t$ by independent increments. These also imply that

$$\sum_{i=1}^{n} a_i B(t_i) = \sum_{i=1}^{n} \Big(\sum_{j=i}^{n} a_j\Big)(B(t_i) - B(t_{i-1})) \quad \text{for } 0 = t_0 \leq t_1 \leq \ldots \leq t_n \text{ and } a_1, \ldots, a_n \in \mathbb{R}$$

is zero-mean normal with variance $\sum_{i=1}^{n}(\sum_{j=i}^{n} a_i)^2 (t_i - t_{i-1})$. □

So we can alternatively define BM as zero-mean Gaussian with covariance function $s \wedge t$.

3.5 Variation and Quadratic variation of BM

Theorem 3.12. $V_B(t) = \infty$ and $[B](t) = t$ for $t > 0$.

Proof. As BM is continuous $[B](t) > 0$ gives $V_B(t) = \infty$ by Theorem 1.4. Now

$$\mathbf{E}\Big\{\sum_{i=1}^{n}(B(t_i) - B(t_{i-1}))^2\Big\} = \sum_{i=1}^{n} \mathbf{E}\big\{(B(t_i) - B(t_{i-1}))^2\big\} = \sum_{i=1}^{n}(t_i - t_{i-1}) = t$$

for $0 = t_0 < t_1 < \ldots < t_n = t$, while

$$\begin{aligned}\mathbf{Var}\Big\{\sum_{i=1}^{n}(B(t_i) - B(t_{i-1}))^2\Big\} &= \sum_{i=1}^{n} \mathbf{Var}\big\{(B(t_i) - B(t_{i-1}))^2\big\} \\ &= \sum_{i=1}^{n} \mathbf{Var}\{\mathrm{N}(0, t_i - t_{i-1})^2\} \\ &= \sum_{i=1}^{n}(t_i - t_{i-1})^2\, \mathbf{Var}\{\mathrm{N}(0,1)^2\} \leq \Big(\max_{1 \leq i \leq n} t_i - t_{i-1}\Big)\, t\, \mathbf{Var}\{\mathrm{N}(0,1)^2\}.\end{aligned}$$ □

3.6 Three Martingales of BM

Theorem 3.13. The following three are martingales wrt. $\{\mathcal{F}_t^B\}_{t\geq 0}$:

1. $\{B(t)\}_{t\geq 0}$,
2. $\{B(t)^2 - t\}_{t\geq 0}$,
3. $\{\mathrm{e}^{uB(t)-u^2t/2}\}_{t\geq 0}$ for any $u\in\mathbb{R}$.

Proof. By independence of increments we have, for $0\leq s\leq t$,

$$\mathbf{E}\{B(t)|\mathcal{F}_s^B\} = \mathbf{E}\{B(t)-B(s)|\mathcal{F}_s^B\} + \mathbf{E}\{B(s)|\mathcal{F}_s^B\} = \mathbf{E}\{B(t)-B(s)\} + B(s) = 0 + B(s),$$

$$\begin{aligned}\mathbf{E}\{B(t)^2-t|\mathcal{F}_s^B\} &= \mathbf{E}\{(B(t)-B(s))^2|\mathcal{F}_s^B\} + \mathbf{E}\{2\,B(s)\,(B(t)-B(s))|\mathcal{F}_s^B\} + \mathbf{E}\{B(s)^2|\mathcal{F}_s^B\} - t\\ &= \mathbf{E}\{(B(t)-B(s))^2\} + 2\,B(s)\,\mathbf{E}\{B(t-B(s)\} + B(s)^2 - t\\ &= (t-s) + 2\cdot B(s)\cdot 0 + B(s)^2 - t,\end{aligned}$$

$$\begin{aligned}\mathbf{E}\{\mathrm{e}^{uB(t)-u^2t/2}|\mathcal{F}_s^B\} &= \mathrm{e}^{uB(s)-u^2t/2}\,\mathbf{E}\{\mathrm{e}^{u(B(t)-B(s))}|\mathcal{F}_s^B\}\\ &= \mathrm{e}^{uB(s)-u^2t/2}\,\mathbf{E}\{\mathrm{e}^{u(B(t)-B(s))}\} = \mathrm{e}^{uB(s)-u^2t/2}\,\mathrm{e}^{u^2(t-s)/2}.\end{aligned}$$

□

3.7 Markov Property of BM

Theorem 3.14. BM is time homogeneous Markov with transition PDF

$$p(t,x,y) = \tfrac{1}{\sqrt{2\pi t}}\,\mathrm{e}^{-(y-x)^2/(2t)} \quad \text{for } t>0 \text{ and } x,y\in\mathbb{R}.$$

Proof. By Theorem 3.6 BM is Markov and by inspection of the proof of Theorem 3.6

$$\begin{aligned}\mathbf{P}\{B(t+s)\in A|B(s)=x\} &= \mathbf{P}\{B(t+s)-B(s)\in A-x\}\\ &= \int_{A-x}\tfrac{1}{\sqrt{2\pi t}}\,\mathrm{e}^{-y^2/(2t)}\,dy = \int_A \tfrac{1}{\sqrt{2\pi t}}\,\mathrm{e}^{-(y-x)^2/(2t)}\,dy.\end{aligned}$$

□