4 Brownian Motion Calculus

An ordinary differential equation (ODE) with initial value is given as

$$x'(t) = \mu(x(t), t) \quad \text{for } t \in [0, T], \quad x(0) = x_0.$$

Equivalently expressed on differential form as

$$dx(t) = \mu(x(t), t)\, dt \quad \text{for } t \in [0, T], \quad x(0) = x_0,$$

and on integrated form as

$$x(t) = x_0 + \int_0^t \mu(x(s), s)\, ds \quad \text{for } t \in [0, T].$$

A stochastic differential equation (SDE) on differential form is given by

$$dX(t) = \mu(X(t), t)\, dt + \sigma(X(t), t)\, dB(t) \quad \text{for } t \in [0, T], \quad X(0) = x_0,$$

where $\{B(t)\}_{t \geq 0}$ is BM. Expressed on integrated form it becomes

$$X(t) = x_0 + \int_0^t \mu(X(s), s)\, ds + \int_0^t \sigma(X(s), s)\, dB(s) \quad \text{for } t \in [0, T].$$

But BM is not FV so that $\int_0^t \ldots dB$ does not exist as a RS-integral and we need to give meaning to the so called Itô integral process

$$\left\{\int_0^t X\, dB\right\}_{t \in [0,T]} = \left\{\int_0^t X(s)\, dB(s)\right\}_{t \in [0,T]}.$$

This integral does not feature in math but is unique for stochastic calculus.

On Convergence of Random Variables

Definition 4.1. A sequence of random variables $\{X_n\}_{n=1}^{\infty}$ converges in probability to X ($X_n \to_{\mathbf{P}} X$) if $\lim_{n\to\infty} \mathbf{P}\{|X_n - X| > \varepsilon\} = 0$ for each $\varepsilon > 0$.

A sequence of random variables $\{X_n\}_{n=1}^{\infty}$ converges in p'th mean, $p \geq 1$, to X ($X_n \to_{\mathbb{L}^p} X$) if $\mathbf{E}\{|X|^p\} < \infty$ and $\lim_{n\to\infty} \mathbf{E}\{(X_n - X)^p\} = 0$.

Theorem 4.2. (Cauchy criterion)

1. X_n converges in probability to some X if and only if $\lim_{m,n\to\infty} \mathbf{P}\{|X_m - X_n| > \varepsilon\} = 0$ for each $\varepsilon > 0$.
2. X_n converges in p'th mean, $p \geq 1$, to some X if and only if $\mathbf{E}\{X_n^p\} < \infty$ for n large enough and $\lim_{m,n\to\infty} \mathbf{E}\{(X_m - X_n)^p\} = 0$.

4.1-4.2 Definition of the Itô Integral and Itô Integral Processes

Henceforth $\{B(t)\}_{t\geq 0}$ is BM and $\{\mathcal{F}_t\}_{\geq 0} = \{\mathcal{F}_t^B\}_{\geq 0}$ the filtration generated by BM itself.

Definition 4.3. A stochastic process $\{X(t)\}_{t\in[0,T]}$ is <u>measurable</u> if a measurable function $X : \Omega \times [0,T] \to \mathbb{R}$, i.e., $X^{-1}(\mathcal{B}) \subseteq \sigma(\mathcal{F} \times ([0,T] \cap \mathcal{B}))$.

The concept of measurable process is too technical to be fully utilized by us so we just inform that processes with cádlág, cáglád or continuous sample paths are measurable.

For a measurable process Fubini's theorem ensures that

$$\mathbf{E}\big\{\int_0^T X(t)\,dt\big\} = \int_0^T \mathbf{E}\{X(t)\}\,dt$$

in the sense that both sides are well-defined simultaneously and when that occur they agree[1].

The Itô integral is done in steps for subsequently larger classes of processes $S_T \subseteq E_T \subseteq P_T$:

Definition 4.4. A measurable and adapted process $\{X(t)\}_{t\in[0,T]}$ is in

- S_T if there exist is a grid $0 = t_0 < t_1 < \ldots < t_n = T$ of (non-random) times and random variables $\xi_0, \xi_1, \ldots, \xi_{n-1}$ with ξ_i $\mathcal{F}_{t_i}$-measurable and $E(\xi_i^2) < \infty$ for $i = 0, \ldots, n-1$ such that
$$X(t) = \xi_0 \mathbf{1}_{\{0\}}(t) + \sum_{i=1}^{n} \xi_{i-1} \mathbf{1}_{(t_{i-1},t_i]}(t) \quad \text{for } t \in [0,T],$$
- E_T if $\mathbf{E}\big\{\int_0^T X(t)^2\,dt\big\} < \infty$,
- P_T if $\mathbf{P}\big\{\int_0^T X(t)^2\,dt < \infty\big\} = 1$.

Definition 4.5. For $X \in S_T$ the <u>Itô integral process</u> is defined

$$\int_0^t X\,dB = \sum_{i=1}^{m} \xi_{i-1}(B(t_i) - B(t_{i-1})) + \xi_m(B(t) - B(t_m))$$

for $t \in (t_m, t_{m+1}]$ and $m = 0, \ldots, n-1$, with $\int_0^0 X\,dB = 0$. Further,

$$\int_s^t X\,dB = \int_0^t X\,dB - \int_0^s X\,dB \quad \text{for } 0 \leq s \leq t.$$

When considering the Itô integral process $\{\int_0^t X\,dB\}_{t\in[0,T]}$ of $X \in S_T$ at times $s_1, \ldots, s_j \in [0,T]$ it is no restriction assume that $s_1, \ldots, s_j$ are members of the grid $0 = t_0 < t_1 < \ldots < t_n = T$ used to define X as otherwise the grid can be enriched to include $s_1, \ldots, s_j$ without affecting values of X or the Itô integral process. This technique is often useful in proofs.

[1] Too measure theoretic to be proved by us.

Theorem 4.6. For $X, Y \in S_T$ we have

1. $\int_0^t (\alpha\, X(s) + \beta\, Y(s))\, dB(s) = \alpha \int_0^t X\, dB + \beta \int_0^t Y\, dB$,
2. $\int_0^t \mathbf{1}_{(a,b]}(s)\, dB(s) = B(b) - B(a)$ for $(a,b] \subseteq [0,t]$,
3. $\int_0^t \mathbf{1}_{(a,b]}(s) X(s)\, dB(s) = \int_a^b X\, dB$ for $(a,b] \subseteq [0,t]$,
4. the Itô integral process $\{\int_0^t X\, dB\}_{t\in[0,T]}$ has continuous sample paths,
5. the Itô integral process $\{\int_0^t X\, dB\}_{t\in[0,T]}$ is adapted to $\{\mathcal{F}_t\}_{t\in[0,T]}$,
6. $[\int_0^t X\, dB] = [\int_0^{(\cdot)} X\, dB](t) = \int_0^t X(s)^2\, ds$ for $t \in [0,T]$,
7. $[\int_0^t X\, dB, \int_0^t Y\, dB] = \int_0^t X(s)Y(s)\, ds$ for $t \in [0,T]$,
8. the Itô integral process $\{\int_0^t X\, dB\}_{t\in[0,T]}$ is a martingale wrt. $\{\mathcal{F}_t\}_{t\in[0,T]}$,
9. $\mathbf{E}\{\int_0^t X\, dB\} = 0$,
10. $\mathbf{E}\big\{(\int_0^t X\, dB)^2\big\} = \mathbf{E}\{\int_0^t X(s)^2\, ds\} = \int_0^t \mathbf{E}\{X(s)^2\}\, ds$ (isometry),
11. $\mathbf{E}\{(\int_0^t X\, dB)\,\{(\int_0^t Y\, dB)\} = \mathbf{E}\{\int_0^t X(s)Y(s)\, ds\} = \int_0^t \mathbf{E}\{X(s)Y(s)\}\, ds$.

Proof. 1-5. By inspection of the definition.

6, 8 and 10. Done in the exercises.

7 and 11. Follows from 6 and 10, respectively by polarization.

9. Follows from 8 as martingales have constant mean. □

Theorem 4.7. For $X \in E_T$ there exists a sequence $\{X_n\}_{n=1}^{\infty} \subseteq S_T$ such that

$$\lim_{n\to\infty} \mathbf{E}\big\{\int_0^T (X_n(t) - X(t))^2\, dt\big\} = 0.$$

Proof. For X continuous[2]*:* Given $\varepsilon > 0$ we need to prove that

$$\mathbf{E}\big\{\int_0^T (Y(t) - X(t))^2\, dt\big\} \le \varepsilon \quad \text{for some } Y \in S_T.$$

To that end let

$$X^{(N)}(t) = \begin{cases} -N & \text{if} \quad X(t) < -N \\ X(t) & \text{if} \quad |X(t)| \le N \\ N & \text{if} \quad X(t) > N \end{cases}.$$

Since $X^{(N)}(t) - X(t) \to 0$ as $N \to \infty$ with $(X^{(N)}(t) - X(t))^2 \le X(t)^2$ we then have

$$\mathbf{E}\big\{\int_0^T (X^{(N)}(t) - X(t))^2\, dt\big\} \to 0 \quad \text{as } N \to \infty$$

[2] The proof for a general not necessarily continuous X is exceptionally difficult.

by dominated convergence as $X \in E_T$. Using the elementary inequality $(x+y)^2 \le 2\,x^2 + 2\,y^2$ it follows that it is enough to prove that, given $\varepsilon > 0$ and $N \in \mathbb{N}$, we have

$$\mathbf{E}\big\{\int_0^T (Y(t) - X^{(N)}(t))^2\,dt\big\} \le \varepsilon \quad \text{for some } Y \in S_T.$$

But as $X^{(N)}$ is uniformly continuous over $[0,T]$ the process

$$Z^{(n)}(t) = \mathbf{1}_{\{0\}}(t)\,X^{(N)}(0) + \sum_{i=1}^{n} \mathbf{1}_{(t_{i-1},t_i]}(t)\,X^{(N)}(t_{i-1}) \quad \text{for } t \in [0,T]$$

in S_T (where $0 = t_0 < t_1 < \ldots < t_n = T$ as usual) satisfies

$$\sup_{t\in[0,T]} \big|Z^{(n)}(t) - X^{(N)}(t)\big| \le \sup_{s,t\in[0,T],\,|s-t|\le \max_{1\le i\le n} t_i - t_{i-1}} \big|X^{(N)}(s) - X^{(N)}(t)\big| \to 0$$

as $\max_{1\le i\le n} t_i - t_{i-1} \downarrow 0$. Hence $Z^{(n)}(t) - X^{(N)}(t) \to 0$ with $(Z^{(n)}(t) - X^{(N)}(t))^2 \le 4\,N^2$, so

$$\mathbf{E}\big\{\int_0^T (Z^{(n)}(t) - X^{(N)}(t))^2\,dt\big\} \to 0 \quad \text{as} \quad \max_{1\le i\le n} t_i - t_{i-1} \downarrow 0$$

by dominated convergence. So we may pick $Y = Z^{(n)}$ with $\max_{1\le i\le n} t_i - t_{i-1}$ small enough.□

Theorem and Definition 4.8. For $X \in E_T$ the Itô integral process $\{\int_0^t X\,dB\}_{t\in[0,T]}$ is well-defined as a mean-square $\to_{\mathbb{L}^2}$ limit of $\int_0^t X_n\,dB$ as $n \to \infty$ for $t \in [0,T]$, where $\{X_n\}_{n=1}^{\infty} \subseteq S_T$ are as in the previous theorem.

Proof. We show that $\{\int_0^t X_n\,dB\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{L}^2$: By isometry for S_T

$$\begin{aligned}
\mathbf{E}\big\{(\textstyle\int_0^t X_n\,dB - \int_0^t X_m\,dB)^2\big\} &= \mathbf{E}\big\{(\textstyle\int_0^t (X_n - X_m)\,dB)^2\big\} \\
&= \mathbf{E}\big\{\textstyle\int_0^t (X_n(t) - X_m(t))^2\,dt\big\} \\
&\le 2\,\mathbf{E}\big\{\textstyle\int_0^t (X_n(t) - X(t))^2\,dt\big\} + 2\,\mathbf{E}\big\{\textstyle\int_0^t (X(t) - X_m(t))^2\,dt\big\} \to 0
\end{aligned}$$

as $m, n \to \infty$. Now, if also $\{\hat{X}_n\}_{n=1}^{\infty} \subseteq S_T$ satisfies

$$\lim_{n\to\infty} \mathbf{E}\big\{\int_0^T (\hat{X}_n(t) - X(t))^2\,dt\big\} = 0,$$

so that $\int_0^t \hat{X}_n\,dB$ converges in mean-square to some limit $\oint_0^t X\,dB$ as $n \to \infty$, we must show that $\int_0^t X\,dB = \oint_0^t X\,dB$. However, this follows from noting that

$$\begin{aligned}
\mathbf{E}\big\{(\textstyle\int_0^t X\,dB - \oint_0^t X\,dB)^2\big\} &\leftarrow \mathbf{E}\big\{(\textstyle\int_0^t X_n\,dB - \int_0^t \hat{X}_n\,dB)^2\big\} \\
&\le 2\,\mathbf{E}\big\{\textstyle\int_0^T (X_n(t) - X(t))^2\,dt\big\} + 2\,\mathbf{E}\big\{\textstyle\int_0^T (X(t) - \hat{X}_n(t))^2\,dt\big\} \to 0.
\end{aligned}$$

Here we used the fact that $Y_n \to_{\mathbb{L}^2} Y$ implies $\mathbf{E}\{Y_n^2\} \to \mathbf{E}\{Y^2\}$ proved in the exercises. □

Theorem 4.9. Properties of the Itô integral process $\{\int_0^t X\,dB\}_{t\in[0,T]}$ for $X \in E_T$ are the same as those for $X \in S_T$ listed in Theorem 4.6.

Proof. 1 and 3. By inspection of the definition.

4 and 6. Too complicated for us to prove.

5. Follows from that limits of measurable functions are measurable.

7-11. As in the proof of Theorem 4.6. □

Theorem 4.10. [3]For $X \in P_T$ there exists $\{X_n\}_{n=1}^{\infty} \subseteq E_T$ such that

$$\int_0^T (X_n(t)-X(t))^2\,dt \to_{\mathbf{P}} 0 \quad \text{as } n\to\infty.$$

Theorem and Definition 4.11. [3]For $X \in P_T$ the Itô integral process $\{\int_0^t X\,dB\}_{t\in[0,T]}$ is well-defined as a convergence in probability $\to_{\mathbf{P}}$ limit of $\int_0^t X_n\,dB$ for $t\in[0,T]$, where $\{X_n\}_{n=1}^{\infty} \subseteq E_T$ are as in the previous theorem.

Theorem 4.12. The properties 1-7 in Theorem 4.6 hold for the Itô integral process $\{\int_0^t X\,dB\}_{t\in[0,T]}$ with $X \in P_T$ while properties 8-11 need not hold.

Proof. 1 and 3-7. As for E_T.

8-9. In the exercises we encounter a non zero-mean Itô integral process.

10-11. For $X \in P_T \setminus E_T$ we have $\int_0^T \mathbf{E}\{X(t)^2\}\,dt = \infty$. □

Theorem 4.13. [3]A continuous and adapted $\{X(t)\}_{t\in[0,T]}$ is in P_T and

$$\sup_{t\in[0,T]}\left|\int_0^t X\,dB - \int_0^t \sum_{i=1}^n X(t_{i-1})\mathbf{1}_{(t_{i-1},t_i]}\,dB\right| \to_{\mathbf{P}} 0$$

when $0=t_0<t_1<\ldots<t_n=T$ with $\max_{1\leq i\leq n} t_i - t_{i-1}\downarrow 0$. In particular

$$\sum_{i=1}^n X(t_{i-1}^n)(B(t_i^n)-B(t_{i-1}^n)) \to_{\mathbf{P}} \int_0^T X\,dB.$$

In general the Itô integral is not an RS integral as

$$\sum_{i=1}^n X(t_i^n)\,(B(t_i^n)-B(t_{i-1}^n)) - \sum_{i=1}^n X(t_{i-1}^n)\,(B(t_i^n)-B(t_{i-1}^n)) \to [X,B](T) \neq 0.$$

But when X is FV we do however have $[X,B](T)=0$ as BM is continuous.

[3]Too technical a result for us to prove.

Example 4.1. (EXAMPLE 4.3 IN KLEBANER) For BM itself we have

$$\sum_{i=1}^{n} B(t_i^n)\,(B(t_i^n)-B(t_{i-1}^n)) - \sum_{i=1}^{n} B(t_{i-1}^n)\,(B(t_i^n)-B(t_{i-1}^n)) \to [B](T)=T.$$

By "twisting" this example a little one deduces that

$$\begin{aligned}
&\textstyle\int_0^T B\,dB \\
&\leftarrow \sum_{i=1}^{n} B(t_{i-1}^n)\,(B(t_i^n)-B(t_{i-1}^n)) \\
&= \tfrac{1}{2}\sum_{i=1}^{n}(B(t_i^n)+B(t_{i-1}^n))\,(B(t_i^n)-B(t_{i-1}^n)) - \tfrac{1}{2}\sum_{i=1}^{n}(B(t_i^n)-B(t_{i-1}^n))\,(B(t_i^n)-B(t_{i-1}^n)) \\
&\to \tfrac{1}{2}\,B(T)^2 - \tfrac{1}{2}\,T.
\end{aligned}$$

4.3 Itô Integral and Gaussian Processes

For a measurable non-random random process $X:[0,T]\to\mathbb{R}$ (with no dependence on $\omega\in\Omega$) we have $X\in E_T$ if and only if $X\in P_T$ if and only if $\int_0^T X(t)^2\,dt<\infty$.

Theorem 4.14. For a non-random $X:[0,T]\to\mathbb{R}$ with $\int_0^T X(t)^2\,dt<\infty$ the Itô integral process $\{\int_0^t X\,dB\}_{t\in[0,T]}$ is zero-mean Gaussian with

$$\mathbf{Cov}\{\textstyle\int_0^s X\,dB, \int_0^t X\,dB\} = \mathbf{E}\{(\int_0^s X\,dB)\,(\int_0^t X\,dB)\} = \int_0^{\min\{s,t\}} X(r)^2\,dr.$$

Proof. Gaussian follows from that limits of Gaussians are Gaussian while zero-mean is by Theorem 4.9. For $0\leq s\leq t$ towering, the martingale property and isometry further give

$$\mathbf{E}\{(\textstyle\int_0^s X\,dB)\,(\int_0^t X\,dB)\} = \mathbf{E}\{(\int_0^s X\,dB)\,\mathbf{E}\{\int_0^t X\,dB|\mathcal{F}_s\}\} = \mathbf{E}\{(\int_0^s X\,dB)^2\} = \int_0^s X(r)^2\,dr. \quad\square$$

Example 4.2. (EXAMPLE 4.10 IN KLEBANER) By Theorem 4.14 $Y(t) = \int_0^t s\,dB(s)$ is $\mathrm{N}(0,t^3/3)$ as $\mathbf{E}\{Y(t)^2\} = \int_0^t s^2\,dt = t^3/3$.

4.4 Itô's Formula for BM

Theorem 4.15. (ITÔ'S FORMULA) For a C^2 function $f:\mathbb{R}\to\mathbb{R}$ we have

$$f(B(t)) = f(B(0)) + \textstyle\int_0^t f'(B(s))\,dB(s) + \frac{1}{2}\int_0^t f''(B(s))\,ds.$$

Itô's formula is often written on differential form as

$$df(B(t)) = f'(B)(t)\,dB(t) + \tfrac{1}{2}\,f''(B(t))\,dt.$$

Proof. With $0 = t_0^n < t_1^n < \ldots < t_n^n = t$ a second order Taylor expansion gives

$$\begin{aligned}
&f(B(t)) - f(B(0)) \\
&= \sum_{i=1}^{n} (f(B(t_i^n)) - f(B(t_{i-1}^n))) \\
&= \sum_{i=1}^{n} f'(B(t_{i-1}^n))\,(B(t_i^n) - B(t_{i-1}^n)) + \tfrac{1}{2}\sum_{i=1}^{n} f''(B(t_{i-1}^n))\,(B(t_i^n) - B(t_{i-1}^n))^2 + \begin{matrix}\text{higher order}\\ \text{terms}\end{matrix}.
\end{aligned}$$

Sending $\max_{1\leq i\leq n} t_i^n - t_{i-1}^n \downarrow 0$ the first term on the right-hand side converges to $\int_0^t f'(B)\,dB$. Being more specific with the third term it is not hard to show that it converges to 0. As for the second sum on the right-hand, that it converges to

$$\int_0^t f''(B(s))\,d[B](s) = \int_0^t f''(B(s))\,ds.$$

follows from that it by continuity of $f''(B)$ is asymptotically the same as

$$\sum_{i=1}^{n} f''(B(s_{j_i}^m))\,(B(t_i^n) - B(t_{i-1}^n))^2$$

for a suitable choice of $0 \leq j_1 \leq \cdots \leq j_n \leq m$, where $0 = s_0^m < s_1^m < \ldots < s_m^m = t$ is a courser grid than $\{t_i^n\}_{i=0}^n$, that is, $m \leq n$ and $\{s_i^m\}_{i=0}^m \subseteq \{t_i^n\}_{i=0}^n$. Now send $\max_{1\leq i\leq n} t_i^n - t_{i-1}^n \downarrow 0$ and $\max_{1\leq i\leq m} s_i^m - s_{i-1}^m \downarrow 0$ afterwards using $[B](t) = t$ and convergence of Riemann sums.□

Corollary 4.16.

1. $dB(t)^2 = d[B](t) = dt$,
2. $dB(t)dt = 0$,
3. $dt^2 = 0$.

Example 4.3. (Examples 4.12-13 in Klebaner) For $f(x) = x^n$ we get

$$\begin{cases} B(t)^n &= n\int_0^t B^{n-1}(s)\,dB(s) + \frac{n(n-1)}{2}\int_0^t B^{n-2}(s)\,ds \\ d(B(t)^n) &= \quad n\,B^{n-1}(t)\,dB(t) + \frac{n(n-1)}{2}\,B^{n-2}(t)\,dt \end{cases}$$

(recovering Example 4.1 with $n = 2$) while $f(x) = \mathrm{e}^x$ gives

$$\begin{cases} \mathrm{e}^{B(t)} &= 1 + \int_0^t \mathrm{e}^{B(s)}\,dB(s) + \frac{1}{2}\int_0^t \mathrm{e}^{B(s)}\,ds \\ d(\mathrm{e}^{B(t)}) &= \quad \mathrm{e}^{B(t)}\,dB(t) + \frac{1}{2}\,\mathrm{e}^{B(t)}\,dt \end{cases}.$$

4.5 Itô Processes and Stochastic Differentials

Definition 4.17. An Itô process is given by

$$\{X(t)\}_{t\in[0,T]} = \left\{X(0) + \int_0^t \mu(s)\,ds + \int_0^t \sigma(s)\,dB(s)\right\}_{t\in[0,T]}$$

where $X(0)$ is a $\mathcal{F}_0$-measurable random variable, μ a measurable and adapted process with $\int_0^T |\mu(t)|\,dt < \infty$ and $\sigma \in P_T$.

A stochastic differential is an Itô process on differential form

$$dX(t) = \mu(t)\,dt + \sigma(t)\,dB(t).$$

Itô processes are adapted and continuous. Further $X(0)$ is a constant as long as $\mathcal{F}_0 = \mathcal{F}_0^B = \{\emptyset, \Omega\}$. Writing $\mu = \mu^+ - \mu^-$ we see that $\{\int_0^t \mu(s)\,ds\}_{t\in[0,T]}$ is FV so that

$$[X](t) = [\int_0^t \mu(s)\,ds + \int_0^t \sigma\,dB, \int_0^t \mu(s)\,ds + \int_0^t \sigma\,dB] = [\int_0^t \sigma\,dB] = \int_0^t \sigma(s)^2\,ds.$$

Using polarization we find the covariation between two Itô processes X and Y

$$[X,Y](t) = \int_0^t \sigma_X(s)\sigma_Y(s)\,ds.$$

Written on differential form this becomes

$$dX(t)dY(t) = d[X,Y](t) = \sigma_X(t)\sigma_Y(t)\,dt.$$

The Itô integral of one Itô process X with respect to another Y is defined as

$$\int_0^t X\,dY = \int_0^t X(s)\,\mu_Y(s)\,ds + \int_0^t X(s)\sigma_Y(s)\,dB(s) \quad \text{for } t\in[0,T]$$

when $\int_0^t |X(s)\mu_Y(s)|\,ds < \infty$ and $X\sigma_Y \in P_T$. Using X's continity it can be shown that

$$\sum_{i=1}^n X(t_{i-1}^n)\,(Y(t_i^n) - Y(t_{i-1}^n)) \to_{\mathbf{P}} \int_0^t X\,dY$$

for partitions $0 = t_0^n < \ldots < t_n^n = t$ of $[0,t]$ such that $\max_{1\le i\le n} t_i^n - t_{i-1}^n \downarrow 0$.

Example 4.3. (Continued) $B(t)^n$ and $\mathrm{e}^{B(t)}$ are Itô processes with stochastic differentials

$$\begin{cases} d(B(t)^n) = \frac{n(n-1)}{2}\,B^{n-2}(t)\,dt + n\,B^{n-1}(t)\,dB(t) \\ d(\mathrm{e}^{B(t)}) = \frac{1}{2}\,\mathrm{e}^{B(t)}\,dt + \mathrm{e}^{B(t)}\,dB(t) \end{cases}.$$

4.6 Itô Formula for Itô processes

By replacing BM with an Itô process X in the derivation of Itô's formula for BM we get

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(s))\,dX(s) + \tfrac{1}{2}\int_0^t f''(X(s))\,d[X](s)$$

for $f : \mathbb{R} \to \mathbb{R}$ that is C^2 on the range of values of X. Written out in full detail this becomes

$$f(X(t)) = f(X(0)) + \int_0^t \big(f'(X(s))\mu_X(s) + \tfrac{1}{2}\, f''(X(s))\sigma_X(s)^2\big)\,ds + \int_0^t f'(X(s))\sigma_X(s)\,dB(s).$$

Writing Itô's formula on differential form we have

$$df(X(t)) = f'(X(t))\,dX(t) + \tfrac{1}{2}\, f''(X(t))\,d[X](t).$$

For two Itô processes X and Y and a C^2 function $f : \mathbb{R}^2 \to \mathbb{R}$ we have

$$\begin{aligned} df(X(t),Y(t)) &= \frac{\partial f}{\partial x}(X(t),Y(t))\,dX(t) + \frac{\partial f}{\partial y}(X(t),Y(t))\,dY(t) \\ &\quad + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(X(t),Y(t))\,d[X](t) + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(X(t),Y(t))\,d[Y](t) \\ &\quad + \frac{\partial^2 f}{\partial x \partial y}(X(t),Y(t))\,d[X,Y](t). \end{aligned}$$

One interesting application of the bivariate Itô formula is integration by parts

$$d(X(t)Y(t)) = X(t)\,dY(t) + Y(t)\,dX(t) + d[X,Y](t).$$

This result can alternatively be established by noting that

$$\begin{aligned} &\sum_{i=1}^{n} (X(t_i^n) - X(t_{i-1}^n))\,(Y(t_i^n) - Y(t_{i-1}^n)) \\ &= X(t)Y(t) - X(0)Y(0) - \sum_{i=1}^{n} X(t_{i-1}^n)\,(Y(t_i^n) - Y(t_{i-1}^n)) - \sum_{i=1}^{n} Y(t_{i-1}^n)\,(X(t_i^n) - X(t_{i-1}^n)) \end{aligned}$$

for $0 = t_0^n < t_1^n < \ldots < t_n^n = t$ and sending $\max_{1 \le i \le n} t_i^n - t_{i-1}^n \downarrow 0$. This integrations by parts formula is not the same as that for the RS integral which do not have the last term on the right-hand side (because it vanishes due to the processes involved being FV and continuous).

Another application of the bivariate Itô formula is the special case when $Y(t) = t$:

$$df(X(t),t) = \frac{\partial f}{\partial x}(X(t),t)\,dX(t) + \frac{\partial f}{\partial t}(X(t),t)\,dt + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(X(t),t)\,d[X](t).$$

Example 4.4. By the previous Itô formula $\{f(B(t),t)\}_{t\in[0,T]}$ is a martingale when $\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} = 0$ and $\{\frac{\partial f}{\partial x}(B(t),t)\}_{t\in[0,T]} \in E_T$. This recovers the three martingales of BM in Chapter 3 by inspection.

Example 4.5. (Example 4.23 in Klebaner) For f a C^2 function we have

$$[f(B),B](t) = \int_0^t f'(B(s))\,ds$$

as by Corollary 4.16 $d[f(B),B](t)$ equals

$$d(f(B(t))dB(t) = (f'(B)(t)\,dB(t) + \tfrac{1}{2}\, f''(B(t))\,dt)\,dB(t) = f'(B)(t)\,dt.$$