Chapter 5 in Klebaner's book

Consider an ODE with initial value

$$x'(t) = \mu(x(t), t) \quad \text{for } t \in [0, T], \quad x(0) = x_0,$$

for a $T \in (0, \infty)$ and an $x_0 \in \mathbb{R}$. According to the Peano theorem it is sufficient to require continuity of the coefficient function $\mu : \mathbb{R} \times [0, T] \to \mathbb{R}$ in order for a solution to exist at least in some sub-interval $[0, S]$ of $[0, T]$. The solution need not be unique.

If the coefficient μ is continuous as well as globally Lipschitz continuous in the second variable, that is,

$$|\mu(t, x) - \mu(t, y)| \leq K\,|x - y| \quad \text{for } t \in [0, T] \text{ and } x, y \in \mathbb{R},$$

for some constant $K > 0$, then the Picard-Lindelöf theorem ensures that there exists a unique solution to the ODE.

We shall now start to discuss stochastic differential equations (SDE)

$$dX(t) = \mu(X(t), t)\, dt + \sigma(X(t), t)\, dB(t) \quad \text{for } t \in [0, T], \quad X(0) = x_0.$$

5.1 Definition of SDE

Through out this chapter $\{B(t)\}_{t\geq 0}$ denotes BM and $\{\mathcal{F}_t\}_{\geq 0} = \{\mathcal{F}_t^B\}_{\geq 0}$ is the filtration generated by BM itself. An SDE with coefficient functions $\mu, \sigma : \mathbb{R} \times [0, T] \to \mathbb{R}$ is given on differential form by

$$dX(t) = \mu(X(t), t)\, dt + \sigma(X(t), t)\, dB(t) \quad \text{for } t \in [0, T], \quad X(0) = x_0.$$

The function μ is called the drift coefficient of the SDE while σ is called the diffusion coefficient. Both these functions must be measurable. The initial value x_0 is a any real (non-random) number. Alternatively, the SDE can be expressed on integrated form as

$$X(t) = x_0 + \int_0^t \mu(X(s), s)\, ds + \int_0^t \sigma(X(s), s)\, dB(s) \quad \text{for } t \in [0, T].$$

A solution X to the SDE is called a diffusion process. Obviously a solution (if it exists) is an Itô process. According to Chapter 4 X must also be adapted and continuous with

$$\int_0^T |\mu(X(t), t)|\, dt < \infty \quad \text{and} \quad \{\sigma(X(t), t)\}_{t\in[0,T]} \in P_T.$$

An SDE can also be started with a random (variable) initial value $X(0) = X_0$ that is independent of $\{\mathcal{F}_t^B\}_{t\geq 0}$. One way to accomplish this is to insert the random X_0 instead of the non-random x_0 in the solution to the non-random initial value SDE when that solution has been constructed. (Similarly to how the abstract conditional expectation can be obtained from the elementary ditto.) This however leads to a non-adapted solution so a more satisfactory way is to work with the enlarged filtration $\mathcal{F}_t = \mathcal{F}_t^B \bigvee \sigma(X_0)$ instead of $\mathcal{F}_t^B$: Everything we have done so far and will do in the sequel based on the filtration being $\mathcal{F}_t^B$ will work without any alterations also for $\mathcal{F}_t^B \bigvee \sigma(X_0)$.

A solution to the SDE is called a strong solution if it is a solution with the BM B given (for any such given B). An SDE has strong uniqueness if (given any BM B) any pair of strong solutions agree (except for on a null-event). Note that this does not assume existence, so an SDE can display strong uniqueness without having a strong solutions. (Such SDE do exists!) Until otherwise is specifically mentioned we discuss and deal with strong solutions only.

One might suspect that a solution to an SDE should resemble a solution of the corresponding ODE with σ taken to be zero. This is sometimes the case but other times the solutions to these SDE and ODE are very different instead.

SDE with coefficients $\mu(x,t) = \mu(x)$ and $\sigma(x,t) = \sigma(x)$ that do not depend on $t \in [0,T]$ are called (time) homogeneous. More detailed results about many issues are known for homogeneous SDE than for general SDE (for example, sharper criteria for existence and/or uniqueness of solutions). We will see a selection of results for homogeneous SDE in Chapter 6. Argubly, most SDE encountered i applied math are homogeneous.

SDE of the above discussed type are the ones we shall focus on and they are more specifically called diffusion type SDE. This is opposed to a more general form of SDE (which we will occasionally also encounter) given by

$$dX(t) = \mu(t)\,dt + \sigma(t)\,dB(t) \quad \text{for } t \in [0,T], \quad X(0) = x_0,$$

where $\mu(t)$ and $\sigma(t)$ may depend on not only $X(t)$ but on the entire past $\{X(s)\}_{s\in[0,t]}$ of X.

There is no general method to find explicit solutions of (diffusion type) SDE expressed in terms of μ, σ and B. On the other hand, when given a candidate to a solution the procedure to validate the solution is often by use of Itô's formula.

Example 5.5 $dX(t) = \mu X(t)\,dt + \sigma X(t)\,dB(t)$, $X(0) = 1$ for $t \geq 0$.

Consider $Y(t) = \ln(X(t))$ so that by Itô's formula

$$dY(t) = \frac{1}{X(t)}\,dX(t) - \frac{1}{2X(t)^2}\,d[X,X](t) =$$

$$= \mu\,dt + \sigma\,dB(t) - \frac{1}{2}\sigma^2\,dt$$

giving $Y(t) = \sigma B(t) + (\mu - \frac{1}{2}\sigma^2)t$ and

$$X(t) = e^{\sigma B(t) + (\mu - \frac{1}{2}\sigma^2)t} \quad \text{for } t \geq 0.$$

Example 5.6 (Langevin equation) $d\mathbb{X}(t) = -\alpha \mathbb{X}(t)dt + \sigma\, dB(t)$
Take $\mathbb{Y}(t) = e^{\alpha t}\mathbb{X}(t)$ (a variant of integrating factor trick from ODE) so by Itô's formula

$$d\mathbb{Y}(t) = \alpha e^{\alpha t}\mathbb{X}(t)\,dt + e^{\alpha t} d\mathbb{X}(t)$$
$$= \alpha \mathbb{Y}(t)\,dt - \alpha \mathbb{Y}(t)\,dt + \sigma e^{\alpha t} dB(t)$$

giving $\mathbb{Y}(t) = X_0 + \int_0^t \sigma e^{\alpha s} dB(s)$
and
$$\mathbb{X}(t) = e^{-\alpha t}\left(X_0 + \int_0^t \sigma e^{\alpha s} dB(s)\right).$$

5.2 Stochastic Exponential and Logarithm

Let X have stochastic differential. A solution $\{U(t)\}_{t\in[0,T]}$ to the (usually non-diffusion type) SDE

$$dU(t) = U(t)\,dX(t) \quad \text{for } t \in [0,T], \quad U(0) = 1,$$

is called a stochastic exponential of X and denoted $\mathcal{E}(X)$. [Note that if the SDE were an ODE the solution would be $U(t) = e^{X(t)-X(0)}$.] It turns out that the stochastic exponential exists and is uniquely given by

$$U(t) = \mathcal{E}(X)(t) = e^{X(t)-X(0)-\frac{1}{2}[X](t)} \quad \text{for } t \in [0,T].$$

Proof $U(t) = f(\mathbb{X}(t) - \mathbb{X}(0), [\mathbb{X},\mathbb{X}](t))$ where $f(x,y) = e^{x-\frac{1}{2}y}$ so

$$dU(t) = U(t)\,d\mathbb{X}(t) - \frac{1}{2}U(t)\,d[\mathbb{X},\mathbb{X}](t)$$
$$+ \frac{1}{2}U(t)\,d[\mathbb{X},\mathbb{X}](t) - \frac{1}{2}U(t)\,d[\mathbb{X},[\mathbb{X},\mathbb{X}]](t) + \frac{1}{8}U(t)\,d[\mathbb{X},\mathbb{X}](t)^2$$
$$= U(t)\,d\mathbb{X}(t) \quad \#$$

Let X have stochastic differential and be strictly positive. A solution $\{U(t)\}_{t\in[0,T]}$ to the (usually non-diffusion type) SDE

$$dX(t) = X(t)\,dU(t) \quad \text{for } t \in [0,T], \quad U(0) = 0,$$

is called a stochastic logarithm of X and denoted $\mathcal{L}(X)$. [Note that if the SDE were an ODE the solution would be $U(t) = \ln(X(t)) - \ln(X(0))$.] It turns out that the stochastic logarithm exists and is uniquely given by

$$U(t) = \mathcal{L}(X)(t) = \ln\left(\frac{X(t)}{X(0)}\right) + \int_0^t \frac{d[X](s)}{2X(s)^2} \quad \text{for } t \in [0,T].$$

A detailed proof of the stochastic logarithm formula is given in the solved exercises.

Example 5.10 We calculate $\mathcal{L}(e^{B(t)})$ using $[e^{B(t)}, e^{B(t)}] = \int_0^t e^{2B(s)}\, ds$:

$$\mathcal{L}(e^{B(t)}) = \ln\left(\frac{e^{B(t)}}{e^{B(0)}}\right) + \int_0^t \frac{e^{2B(s)}\, ds}{2e^{2B(s)}} = B(t) - B(0) + t/2 \quad \#$$

It is a very useful exercise to establish that $\mathcal{L}(\mathcal{E}(X)) = X - X(0)$ and $\mathcal{E}(\mathcal{L}(X)) = X/X(0)$.

5.3 Solutions to Linear SDE

A general linear (in general non-diffusion type) SDE is given by

$$dX(t) = (\alpha(t) + \beta(t)X(t))\, dt + (\gamma(t) + \delta(t)X(t))\, dB(t) \quad \text{for } t \in [0,T], \quad X(0) = x_0.$$

Here the coefficients α, β, γ and δ are continuous adapted stochastic processes satisfying appropriate integrability conditions. It is not very hard (albeit a bit notationally complicated) to show that the solution to the general linear SDE is given by

$$X(t) = U(t)\left(x_0 + \int_0^t \frac{\alpha(s) - \gamma(s)\delta(s)}{U(s)}\, ds + \int_0^t \frac{\gamma}{U}\, dB\right)$$

with

$$U(t) = \exp\left(\int_0^t (\beta(s) - \tfrac{1}{2}\delta(s)^2)\, ds + \int_0^t \delta\, dB\right) \quad \text{for } t \in [0,T].$$

One famous example of a linear SDE is the Langevin equation $\alpha(t) = -\mu$, $\beta = \delta = 0$ and $\sigma(t) = \sigma$ for constants $\mu, \sigma > 0$. Another is the stochastic exponent of BM $\alpha = \beta = \sigma = 0$ and $\delta(t) = 1$. Yet another example is offered by the so called Brownian bridge

$$dX(t) = \frac{b - X(t)}{T - t}\, dt + dB(t) \quad \text{for } t \in [0,T), \quad X(0) = a.$$

This turns out to simply be the process $X(t) = (B(t) + a | B(T) + a = b)$, that is, Brownian motion on $[0,T]$ started at a and forced (or conditioned rather) to finish at b.

5.4 Existence and Uniqueness of Solutions

We will cite two basic results for existence and/or uniqueness of solutions to SDE. They can both be somewhat strengthened in different ways at the cost of a more complicated apperance. We do not discuss such improvements.

5.5

The first result is the basic existence and uniqueness result for an SDE

$$dX(t) = \mu(X(t), t)\,dt + \sigma(X(t), t)\,dB(t) \quad \text{for } t \in [0, T]:$$

Assume that there exist constants $K_1 = K_1(N, T) > 0$ and $K_2 = K_2(T) > 0$ such that

$$|\mu(x,t) - \mu(y,t)| + |\sigma(x,t) - \sigma(y,t)| \le K_1|x - y| \quad \text{for } t \in [0,T] \text{ and } |x|, |y| \le N$$

for each $N > 0$ and

$$|\mu(x,t)| + |\sigma(x,t)| \le K_2(1 + |x|) \quad \text{for } t \in [0,T] \text{ and } x \in \mathbb{R}.$$

In other words the coefficients are locally Lipschitz with (global) linear growth in the x-variable uniformly in the t-variable. For any choiche of a non-random initial value $X(0) = x_0$ or a random initial value $X(0) = X_0$ there exists a unique strong solution $\{X(t)\}_{t\in[0,T]}$ to the SDE. If in addition X_0 satisfies $E(X_0^2) < \infty$ then it holds that

$$E\Big(\sup_{t\in[0,T]} X(t)^2\Big) \le C(1 + E(X_0^2)),$$

where $C = C(K_2, T)$ is a constant that depends on K_2 and T only.

The local Lipschitz condition suffices for uniqueness, but not for existence as one can see, for example, with the ODE $dx(t) = x(t)^2\,dt$, $x(0) = 1$, with unique solution $x(t) = 1/(1-t)$. It is easy to see that a global Lipschitz condition, that is, $K_1 = K_1(T)$ not depending on N, implies linear growth.

Obviously the canonical application of the above cited existence and uniqueness result is to a linear SDE with bounded non-random (to make the SDE diffusion type) coefficients $\alpha, \beta, \sigma, \delta : [0,T] \to \mathbb{R}$.

The second basic result only concerns uniqueness and is attributed to Yamada and Watanabe: Assume that the drift coefficient μ is globally Lipschitz and that the diffusion coefficient σ is globally Hölder of order $\alpha \ge 1/2$, that is,

$$|\sigma(x,t) - \sigma(y,t)| \le K_3|x - y|^\alpha \quad \text{for } t \in [0,T] \text{ and } x, y \in \mathbb{R}$$

for some constant $K_3 = K_3(T)$. Then the SDE displays strong uniqueness.

The canonical application of the Yamada-Watanabe theorem is to the so called Girsanov SDE $dX(t) = |X(t)|^r\,dB(t)$, $X(0) = 0$, with $r \ge 1/2$ for which the theorem yields uniqueness of the solution $X = 0$.

The criteria/ imposed in the above cited existence and uniqueness results are very far from necessary: SDE can have very "wild" coefficients [such as, for example, $\sigma(x,t) = |x|^{1000}$] and still have a well-defined unique solution.

5.5 Markov Property of Solutions

More or less in general, solutions to SDE are Markov processes.

If the SDE is time homogeneous then so is the Markov process solution. In fact, at least historically, some people called solutions to SDE (diffusion processes) continuous Markov processes. This indicates that the converse statement that continuous Markov processes are diffusion processes hold. However we will not discuss this topic.

5.6 Construction of Weak Solutions

5.7

A solution to the SDE is called a weak solution if the BM that features in (the solution to) the SDE is (not given from the beginning but) constructed together with the solution (can be chosen at liberty). Clearly any strong solution is a weak solution.

An SDE has weak uniqueness if any pair of weak solutions $\{X_1(t)\}_{t\in[0,T]}$ and $\{X_2(t)\}_{t\in[0,T]}$ have common finite dimensional distributions, that is,

$$P(X_1(t_1) \leq x_1, \ldots, X_1(t_n) \leq x_n) = P(X_2(t_1) \leq x_1, \ldots, X_2(t_n) \leq x_n)$$

for $x_1, \ldots, x_n \in \mathbb{R}$, $t_1, \ldots, t_n \in [0, T]$ and $n \in \mathbb{N}$.

The difference between strong and weak solutions might seem minor but in fact it is the other way around: It is from a theoretical perspective substantially easier to find weak solutions than to find strong solutions. However, for virtually any specific SDE encountered strong solutions exist when weak solutions do.

There is a famous example of an SDE that has a unique weak solution but no strong solution, namely the Tanaka SDE $dX(t) = \text{sign}(X(t))\,dB(t)$, $X(0) = 0$. [Here $\text{sign}(0) = 1$ as otherwise $X(0) = 0$ would be a strong solution.] The statement about the weak solution is not too hard to prove while that for (no) strong solutions requires a quite sophisticated extension of Itô's formula to the convex function $\mathbb{R} \ni x \curvearrowright |x| \in \mathbb{R}$.

Example 5.15 (Tanaka SDE) $dX(t) = \text{sign}(X(t))\,dB(t)$.
Let $\hat{B}(t) = \int_0^t \frac{1}{\text{sign}(X(s))}\,dX(s) = \int_0^t \text{sign}(X(s))\,dX(s)$ for some BM $X(t)$. As $1/\text{sign}(X) = \text{sign}(X) \in E_T$ $\hat{B}$ is a continuous martingale with quadratic variation $[\hat{B}, \hat{B}] = \int_0^t \text{sign}(X(s))^2\,ds = t$ and must therefore be a BM (according to the so called Lévy characterization). As $d\hat{B}(t) = \frac{1}{\text{sign}(X(t))}\,dX(t)$ gives $dX(t) = \text{sign}(X(t))\,d\hat{B}(t)$ we have created a weak solution to Tanaka's SDE.

The basic existence and uniqueness result for weak solutions to an SDE

$$dX(t) = \mu(X(t),t)\,dt + \sigma(X(t),t)\,dB(t) \quad \text{for } t \in [0,T], \quad X(0) = x_0,$$

is as follows: Assume that $\sigma(x,t)$ is strictly positive and continuous and that both $\mu(x,t)$ and $\sigma(x,t)$ have linear growth in the x-variable (uniformly in the t-variable). Then the SDE has a unique solution, ~~that is a strong Markov process~~. Note that the Lipschitz condition for the coefficients featuring in the basic existence and uniqueness result for strong solutions is not needed for weak solutions.

There are many connections between solutions to SDE and solutions to partial differential equation (PDE) as we shall see. The generator of an SDE is the partial differential operator

$$L_s f(x,s) = (L_s f)(x,s) = \frac{\sigma(x,s)^2}{2}\,\frac{\partial^2 f(x,s)}{\partial x^2} + \mu(x,s)\,\frac{\partial f(x,s)}{\partial x}.$$

Now assume that μ and σ ~~is~~ are bounded on compact subsets of ~~$[0,T]\times\mathbb{R}$~~ $\mathbb{R}\times[0,T]$. From Itô's formula

$$df(X(t),t) = f'_x(X(t),t)\,dX(t) + \tfrac{1}{2} f''_{xx}(X(t),t)\,d[X,X](t) + f'_t(X(t),t)\,dt$$
$$= \underline{f'(X(t),t)\,\sigma(X(t),t)\,dB(t)} + \Big[f'_x(X(t),t)\,\mu(X(t),t) + \tfrac{1}{2} f''_{xx}(X(t),t)\,\sigma(X(t),t)^2 + f'(X(t),t)\Big]\,dt$$

so that

$$\Big\{ f(X(t),t) - f(x_0,0) - \int_0^t \Big(L_s f(X(s),s) + \frac{\partial f(X(s),s)}{\partial s}\Big)\,ds \Big\}_{t\geq 0}$$

as equal this Itô integral

is a zero-mean martingale for $f:\mathbb{R}\times[0,T]\to\mathbb{R}$ C^2 ~~with bounded support~~

Note ~~that~~: $f(X(t),t)$ mart if $L_s f + \frac{\partial f}{\partial s} = 0$

Given just the coefficients μ and σ of the SDE together with the initial value x_0, a martingale problem for the SDE is the converse issue to find a continuous and adapted stochastic process $\{X(t)\}_{t\in[0,T]}$ such that the above process is a zero-mean martingale for all C^2-functions f with bounded suppor

Example As BM B has generator $L_t f(x,t) = \frac{1}{2}\frac{\partial^2 f(x,t)}{\partial x^2}$ we have that $f(B(t),t) - f(0,0)$ as well as $f(B(t),t)$ are martingales as soon as $(\cancel{L_t} + \frac{\partial}{\partial t}) f(x,t) = \frac{1}{2}\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial t} = 0$. This can recover all three examples of martingales of BM we saw in Chapter 3 $B(t)$, $B(t)^2 - t$ and $e^{B(t)-t/2}$.

5.8

5.~~7~~ 8 Backward and Forward Equations

Let L_s be the generator of the SDE

$$dX(t) = \mu(X(t),t)\,dt + \sigma(X(t),t)\,dB(t) \quad \text{for } t \in [0,T].$$

A fundamental solution $p : \mathbb{R} \times [0,T] \to \mathbb{R}$ of the so called Kolmogorov backward PDE

$$\frac{\partial p(x,s)}{\partial s} + L_s p(x,s) = 0$$

is a non-negative function $p(t,y,x,s)$ for $x,y \in \mathbb{R}$ and $0 \le s < t \le T$ such that

$$u(x,s) = \int_{-\infty}^{\infty} g(y)p(y,t,x,s)\,dy$$

is bounded and [regarded as a function of (x,s)] solves the PDE for any bounded function $g : \mathbb{R} \to \mathbb{R}$ with

$$\lim_{s\uparrow t} u(x,s) = g(x).$$

Under technical conditions on the coefficients μ and σ a strictly positive fundamental solution $p(y,t,x,s)$ to the backward equation exists. ~~Further~~ That fundamental solution solves the backward equation [regarded as a function of (x,s)] as well as the so called forward equation

$$-\frac{\partial p}{\partial t} + L_t^\star p = -\frac{\partial p}{\partial t} + \frac{\partial^2}{\partial y^2}\left(\frac{\sigma(y,t)^2}{2}\,p\right) - \frac{\partial}{\partial y}\big(\mu(y,t)\,p\big) = 0$$

[regarded as a function of (y,t)]. (Here the $\star$ in the notation stems from that $L^\star$ is the adjoint differential operator to L in a certain sense.) Moreover there exists a Markov process $\{X(t)\}_{t\in[0,T]}$ that has $p(y,t,x,s)$ as transition PDF and is a weak solution to the SDE. (This of course requires a construction of a BM B associated with the Markov process X such that X solves the SDE.)

And so we have found a method to construct weak solutions to SDE based on theory for PDE and Markov process.

5.~~8~~ 9 Stratonovich Stochastic Calculus

There is an alternative to the so called Itô stochastic calculus we have developed so far labeled Stratonovich stochastic calculus. The idea here is to alter the definition of the stochastic integral involved so that the rules from ordinary calculus (such as, for example, integration by parts) come into play again. The underlying theory still is the same though, as are the Itô processes involved, being the sum of an Itô integral process and an (absolutely) continuous FV process. It is just a matter of expessing them differently.

The Stratonovich integral process $\{\int_0^t X\,\partial Y\}_{t\in[0,T]}$ of an Itô process $\{X(t)\}_{t\in[0,T]}$ with respect to another Itô process $\{Y(t)\}_{t\in[0,T]}$ (both built with and adapted to the filtration of the one and same BM B) is defined as

$$\left\{\int_0^t X\,\partial Y\right\}_{t\in[0,T]} = \left\{\int_0^t X\,dY + \frac{1}{2}[X,Y](t)\right\}_{t\in[0,T]}.$$

Written on differential form this becomes

$$X(t)\,\partial Y(t) = X(t)\,dY(t) + \frac{1}{2}\,d[X,Y](t) \quad \text{for } t\in[0,T].$$

As X is continuous it follows from what we learned about Itô integrals of one Itô process with respecect to another together with the definition of variation that

$$\frac{1}{2}\sum_{i=1}^{n}(X(t_i^n)+X(t_{i-1}^n))(Y(t_i^n)-Y(t_{i-1}^n)) \to_P \int_0^t X\,\partial Y$$

for a sequence of partitions $0=t_0^n<\ldots<t_n^n=t$ of $[0,t]$ such that $\max_{1\le i\le n} t_i^n - t_{i-1}^n \downarrow 0$.

Expressed with Stratonovich differentials the chain rule becomes

$$d(X(t)Y(t)) = X(t)\,dY(t) + Y(t)\,dX(t) + d[X,Y](t) = X(t)\,\partial Y(t) + Y(t)\,\partial X(t).$$

Making use of the fact that

$$d[f'(X)(t),X(t)] = \left(f''(X(t))\,dX(t) + \tfrac{1}{2}f'''(X(t))\,d[X](t)\right)dX(t) = f''(X(t))\,d[X](t)$$

for $f:\mathbb{R}\to\mathbb{R}$ of class C^3 it further follows that Itô's formula (for $f\in C^3$) simplifies to

$$df(X(t)) = f'(X(t))\,dX(t) + \tfrac{1}{2}f''(X(t))\,d[X](t) = f'(X(t))\,\partial X(t).$$

By mere insertion in the definition on differential form of a Stratonovich (integral) diffusion type SDE

$$dX(t) = \mu(X(t),t)\,dt + \sigma(X(t),t)\,\partial B(t) \quad \text{for } t\in[0,T]$$

and using that

$$d[\sigma(X(t),t),B(t)] = \sigma(X(t),t)\,\frac{\partial\sigma}{\partial x}(X(t),t)\;dt$$

we see that the equation translates to the Itô (integral) SDE

$$dX(t) = \left(\mu(X(t),t) + \frac{1}{2}\sigma(X(t),t)\,\frac{\partial\sigma}{\partial x}(X(t),t)\right)dt + \sigma(X(t),t)\,dB(t) \quad \text{for } t\in[0,T].$$