6 Diffusion Processes

6.1 Martingales and Dynkin's Formula

Consider a general SDE of diffusion type

$$dX(t) = \mu(X(t),t)\,dt + \sigma(X(t),t)\,dB(t) \quad \text{for } t \in [0,T]$$

with generator

$$L_t f(x,t) = (L_t f)(x,t) = \frac{\sigma(x,t)^2}{2}\,\frac{\partial^2 f(x,t)}{\partial x^2} + \mu(x,t)\,\frac{\partial f(x,t)}{\partial x}.$$

Itô's formula for $f(X(t),t)$ with $f \in C^{2,1}$ (when X solves the SDE) can be expressed with help of the generator as

$$df(X(t),t) = \left(L_t f(X(t),t) + \frac{\partial f}{\partial t}(X(t),t)\right)dt + \left(\frac{\partial f}{\partial x}(X(t),t)\right)\sigma(X(t),t)\,dB(t).$$

And so it follows that under suitable technical conditions the following process is a martingale:

$$\left\{ f(X(t),t) - \int_0^t \left(L_s f + \frac{\partial f}{\partial s}\right)(X(s),s)\,ds \right\}_{t\in[0,T]}.$$

As an immediate collorary to the previous paragraph it follows that (under technical conditions) $\{f(X(t),t)\}_{t\in[0,T]}$ is a martingale if f solves the backward equation $(L_t + \frac{\partial}{\partial t})f(x,t) = 0$. Although just a way of rewriting things we already know this observation can be surprisingly useful.

Just by taking expectations of what we have found out above, it follows that if X is started deterministically $X(0) = x_0$ it holds for $f \in C^{2,1}$ (under technical conditions) that

$$E(f(X(t),t)) = f(x_0,0) + E\left(\int_0^t \left(L_s f + \frac{\partial f}{\partial s}\right)(X(s),s)\,ds\right) \quad \text{for } t \in [0,T].$$

This simple result can extended to $t = \tau$ where τ is a bounded stopping time with $\tau \in [0,T]$. This extended version of the result is called Dynkin's fomula (after the important contributer to SDE theory E.B. Dynkin)

6.2 Calculation of Expectations and PDE

Consider a general SDE of diffusion type

$$dX(t) = \mu(X(t),t)\,dt + \sigma(X(t),t)\,dB(t) \quad \text{for } t \in [0,T]$$

with generator

$$L_t f(x,t) = (L_t f)(x,t) = \frac{\sigma(x,t)^2}{2}\,\frac{\partial^2 f(x,t)}{\partial x^2} + \mu(x,t)\,\frac{\partial f(x,t)}{\partial x}.$$

Under technical conditions a solution to the backward equation terminal boundary problem

$$\left(L_t + \frac{\partial}{\partial t}\right) f(x,t) = 0 \quad \text{for } t \in [0,T], \quad f(x,T) = g(x),$$

(if it exists) must be given by

alternatively $$f(x,t) = E(g(X(T))|X(t) = x) \quad \text{for } t \in [0,T],$$

where X solves the SDE. (Just time shift the coefficients to get an SDE that starts at time 0 instead of time t if you want to.) As the backward equation ensures that $\{f(X(s),s)\}_{s\in[t,T]}$ is a martingale we have

$$E(g(X(T))|\mathcal{F}_t) = E(f(X(T),T)|\mathcal{F}_t) = f(X(t),t) \quad \text{for } t \in [0,T].$$

Now apply the Markov property to the left-hand side to obtain

$$E(g(X(T))|X(t) = x) = f(x,t) \quad \text{for } t \in [0,T].$$

Conversely $f(x,t) = E(g(X(T))|X(t) = x)$ solves the backward equation terminal value problem.

$f(x,t) = E(f(X(t),t)|X(t)=x) \overset{\text{towering}}{=} E(E(f(X(t),t)|\mathcal{F}_t)|X(t)=x) \overset{f(X(t),t)\ \text{martingale}}{=} E(E(f(X(T),T)|\mathcal{F}_t)|X(t)=x) = E(E(g(X(T))|\mathcal{F}_t)|X(t)=x) \overset{\text{towering}}{=} E(g(X(T))|X(t)=x)$

There exist several generalisations of the first of the above mentioned basic relation between solution to the SDE and to the corresponding backward equation out of which we will mention two. (The proofs are similar to the proof we did above but just notationally slightly more complicated.)

- Under technical condition, the solution to the terminal boundary problem

$$\left(L_t + \frac{\partial}{\partial t}\right) f(x,t) = \phi(x) \quad \text{for } t \in [0,T], \quad f(x,T) = g(x),$$

(if it exists) must be given by

$$f(x,t) = E\left(g(X(T)) - \int_t^T \phi(X(s))\,ds \,\middle|\, X(t) = x\right) \quad \text{for } t \in [0,T],$$

where X solves the SDE.

- (Feynman-Kac Formula) Under technical condition, the solution to the terminal boundary problem

$$\left(L_t + \frac{\partial}{\partial t}\right) f(x,t) = r(x,t) f(x,t) \quad \text{for } t \in [0,T], \quad f(x,T) = g(x),$$

(if it exists) must be given by

$$f(x,t) = E\left(g(X(T)) \exp\left(-\int_t^T r(X(s),s)\,ds\right) \,\middle|\, X(t) = x\right) \quad \text{for } t \in [0,T],$$

where X solves the SDE.

6.3

Example 6.5 $\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} + \mu x \frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} = rf, \quad f(x,T) = x^2$

$\sigma(x,t) = \sigma x, \quad \mu(x,t) = \mu x, \quad r(x,t) = r, \quad g(x) = x^2$

$dX(t) = \mu X(t)\,dt + \sigma X(t)\,dB(t)$

$X(t) = X(0)\, e^{(\mu - \sigma^2/2)t}\, e^{\sigma B(t)} \Rightarrow X(T) = X(t)\, e^{(\mu - \sigma^2/2)(T-t)}\, e^{\sigma(B(T) - B(t))}$

$f(x,t) = E\left(g(X(T)) \exp\left(-\int_t^T r(X(s),s)\,ds\right) \,\middle|\, X(t) = x\right)$

$= E\left(X(T)^2 e^{-r(T-t)} \,\middle|\, X(t) = x\right)$

$= e^{-r(T-t)} E\left(x^2 e^{(2\mu - \sigma^2)(T-t)} e^{2\sigma(B(T) - B(t))}\right)$

$= e^{-r(T-t)} x^2 e^{(2\mu - \sigma^2)(T-t)} e^{2\sigma^2(T-t)} = x^2 e^{(2\mu + \sigma^2 - r)(T-t)}$

6.3 Time Homogeneous Diffusions

Consider a time homogeneous SDE of diffusion type

$$dX(t) = \mu(X(t))\,dt + \sigma(X(t))\,dB(t) \quad \text{for } t \in [0,T].$$

Because of (time) homogeneity the generator now simplifies to

$$Lf(x) = (Lf)(x) = \frac{\sigma(x)^2}{2} f''(x) + \mu(x) f'(x).$$

Itô's formula for $f(X(t))$ with $f \in C^2$ (when X solves the SDE) can be expressed with help of the generator as

$$df(X(t)) = Lf(X(t))\,dt + f'(X(t))\,dB(t) \quad \text{for } t \in [0,T].$$

It follows that under suitable technical conditions the following process is a martingale:

$$\left\{ f(X(t)) - \int_0^t Lf(X(s))\,ds \right\}_{t \in [0,T]}.$$

Weak solutions can be found by solving the corresponding martingale problem (to find an X that makes this process a martingale for a suitable class of functions $f \in C^2$).

The existence and uniqueness criteria for weak solutions that are strong Markov is as before with the t parameter of the coefficients $\mu(x,t)$ and $\sigma(x,t)$ removed. Further it follows from applying time shifts that if there exists a unique weak solution for every starting value $X(0) = x$, then the transition CDF will be homogeneous

$$P(y,t,x,s) = P(y,t-s,x,0) = P(t-s,x,y) \quad \text{for } t \in (s,T].$$

Of course, then also the corresponding transition PDF $p(y,t,x,s) = p(t-s,x,y)$ will be homogeneous if it exists (which it usually does).

Under appropriate technical conditions it follows from what we did for general (not necessarily time homogeneous) SDE that $p(t,x,y)$ satisfies the Kolmogorov backward equation

$$\frac{\partial p(t,x,y)}{\partial t} - Lp(t,x,y) = \frac{\partial p(t,x,y)}{\partial t} - \frac{\sigma(x)^2}{2}\frac{\partial^2 p(t,x,y)}{\partial x^2} - \mu(x)\frac{\partial p(t,x,y)}{\partial x} = 0.$$

Note that the time variable t now handles both the backward time variable s and the forward time variable t of $p(y,t,x,s)$ so that the sign of the time derivative changes. The corresponding Kolmogorov forward equation becomes

$$\frac{\partial p(t,x,y)}{\partial t} - L^\star p(t,x,y) = \frac{\partial p(t,x,y)}{\partial t} - \frac{\partial^2}{\partial y^2}\left(\frac{\sigma(y)^2}{2}p(t,x,y)\right) + \frac{\partial}{\partial y}(\mu(y)p(t,x,y) = 0.$$

There is a famous result that completely resolves the existence and uniqueness issues for a homogeneous SDE with zero drift due to Engelbert and Schmidt: The SDE

$$dX(t) = \sigma(X(t))\,dB(t) \quad \text{for } t \in [0,T], \quad X(0) = x_0,$$

has a non-exploding (see a later section for explanation of this term) weak solution for every initial value x_0 if and only for every x it holds that

$$\int_{-a}^{a} \frac{dy}{\sigma(x+y)^2} = \infty \quad \text{for } a > 0 \;\Rightarrow\; \sigma(x) = 0.$$

Moreover the SDE has a unique (non-exploding) weak solution for every initial value x_0 if and only for every x it holds that

$$\int_{-a}^{a} \frac{dy}{\sigma(x+y)^2} = \infty \quad \text{for } a > 0 \;\Leftrightarrow\; \sigma(x) = 0.$$

These results have no counterpart for non-homogeneous SDE. However, by means of application of Itô's formula conclusions can also be drawn about homogeneuous SDE with non-zero drift. We will see how a result for SDE with zero drift can be carried over to non-zero drift SDE in the next section.

Example 6.8 The Tanaka SDE $dX(t) = \text{sign}(X(t))\,dB(t)$, $X(0)=0$ has a unique weak solution.

Example 6.9 Every SDE $dX(t) = \sigma(X(t))\,dB(t)$, $X(0)=x_0$ with σ strictly positive continuous has a unique weak solution. And if σ is continuous it has a weak solution (not necessarily unique).

6.5

6.5 Explosion

Let $\{X(t)\}_{t\geq 0}$ solve the homogeneous SDE

$$dX(t) = \mu(X(t))\,dt + \sigma(X(t))\,dB(t) \quad \text{for } t \geq 0, \quad X(0) = x.$$

For simpler notation we also assume that the values of X are not restricted to any finite or half finite interval $(\alpha, \beta) \subseteq \mathbb{R}$. However, the things we do under the latter assumption can be extended in a more or less obvious way to the general case.

Let $\tau_n = \inf\{t \geq 0 : |X(t)| = n\}$. Clearly the limit $\lim_{n\to\infty} \tau_n = \tau_\infty$ exists (although possibly $\tau_\infty = \infty$). We say that X explodes if $P(\tau_\infty < \infty) > 0$. Note that on the event $\{\tau_\infty < \infty\}$ we have $|X(\tau_\infty)| = \lim_{n\to\infty} |X(\tau_n)| = \infty$ which motivates the preceding language.

Clearly, under appropriate conditions on μ and σ (making no move of X impossible), X explodes started at a any $x \in \mathbb{R}$ if X explodes started at a particular x.

Now assume that σ is strictly positive and continuous and that μ is bounded on finite intervals (which are sufficient appropriate conditions in the previous paragraph). Given any $x_0 \in \mathbb{R}$, the diffusion X explodes when started at a particular $x \in \mathbb{R}$ if and only if X explodes started at any $x \in \mathbb{R}$ if and only if at least one of the following two integrals are finite

$$\int_{-\infty}^{x_0} \exp\Big(-\int_{x_0}^{x} \frac{2\mu(y)}{\sigma(y)^2}\,dy\Big)\Big(\int_{x}^{x_0} \frac{1}{\sigma(y)^2} \exp\Big(\int_{x_0}^{y} \frac{2\mu(z)}{\sigma(z)^2}\,dz\Big)dy\Big)dx$$

and

$$\int_{x_0}^{\infty} \exp\Big(-\int_{x_0}^{x} \frac{2\mu(y)}{\sigma(y)^2}\,dy\Big)\Big(\int_{x_0}^{x} \frac{1}{\sigma(y)^2} \exp\Big(\int_{x_0}^{y} \frac{2\mu(z)}{\sigma(z)^2}\,dz\Big)dy\Big)dx.$$

Because of the many integrals (primitive functions) involved and that it often is impossible to judge just by inspection if the above two integrals are finite or not, it can be a quite cumbersome task to check whether the above two integrals are finite or not.

The ODE $dx(t) = x(t)^2\,dt$, $x(0) = 1$, with solution $x(t) = 1/(1-t)$ for $t \in [0, 1)$ explodes. No SDE with zero drift (and strictly positive continuous σ) explodes.

6.6 Reccurence and Transience

Let $\{X(t)\}_{t\geq 0}$ solve the homogeneous SDE

$$dX(t) = \mu(X(t))\,dt + \sigma(X(t))\,dB(t) \quad \text{for } t \geq 0, \quad X(0) = x.$$

For simpler notation we also assume that the values of X are not restricted to any finite or half finite interval $(\alpha, \beta) \subseteq \mathbb{R}$.

The starting point $x \in \mathbb{R}$ i called reccurent if $X(t_i) = x$ for a sequence of (usually random) times $0 < t_0 < t_1 < \ldots$ such that $t_n \to \infty$ as $n \to \infty$ with probability 1. If all starting points are reccurent then X is called reccurent. [To require that X visits x infinitely many times is not the same thing because the (usually) infinite variation of X can give infinitely many visits in a bounded interval.]

The starting point $x \in \mathbb{R}$ is called transient if $|X(t)| \to \infty$ as $t \to \infty$ with probability 1. If all starting points are transient then X is called transient.

We now make the same assumption that were used in the existence and uniqueness criteria for weak solutions that are strong Markov processes to the SDE: Let σ be strictly positive and continuous and let both μ and σ have linear growth. Then if there is one reccurent starting point X then the diffusion is reccurent. Further, if there are no reccurent starting points, then the diffusion is transient. Moreover, given any $x_0 \in \mathbb{R}$, the diffusion is reccurent if and only if the following two integrals both are infinite

$$\int_{-\infty}^{x_0} \exp\left(-\int_{x_0}^{x} \frac{2\mu(y)}{\sigma(y)^2}\,dy\right) dx \quad \text{and} \quad \int_{x_0}^{\infty} \exp\left(-\int_{x_0}^{x} \frac{2\mu(y)}{\sigma(y)^2}\,dy\right) dx.$$

Example $dX(t) = \alpha X(t)\,dt + \sigma\,dB(t)$, $X_0 = 0$, $\mu(x,t) = \alpha x$, $\sigma(x,t) = \sigma$

$$\int_0^x \frac{2\mu(y)}{\sigma(y)^2}\,dy = \frac{\alpha x^2}{\sigma^2}, \quad \int_{-\infty}^{0} e^{-\frac{\alpha x^2}{\sigma^2}}\,dx = \int_0^{+\infty} e^{-\frac{\alpha x^2}{\sigma^2}}\,dx = \begin{cases} +\infty & \text{for } \alpha \le 0 \\ < \infty & \text{for } \alpha > 0 \end{cases}$$

So reccurent for $\alpha \le 0$. In particular the Langevin Equation and BM are reccurent.

6.8 Stationary Distributions

Let $\{X(t)\}_{t \ge 0}$ solve the homogeneous SDE

$$dX(t) = \mu(X(t))\,dt + \sigma(X(t))\,dB(t) \quad \text{for } t \ge 0, \quad X(0) = X_0.$$

Assume that X is a Markov process with transition CDF $P(t,x,y)$. For simpler notation we also assume that the values of X are not restricted to any finite or half finite interval $(\alpha, \beta) \subseteq \mathbb{R}$.

A CDF $\Pi : \mathbb{R} \to [0,1]$ is called a stationary CDF or an invariant CDF for X if

$$\Pi(y) = \int_{-\infty}^{\infty} P(t,x,y)\,d\Pi(x) \quad \text{for } y \in \mathbb{R}.$$

This means that if $X(s)$ has CDF Π for an $s \ge 0$, then also $X(t+s)$ has CDF Π for $t > 0$.

Assume that the transition CDF $P(t,x,y)$ has a transition PDF $p(t,x,y) = \frac{\partial}{\partial y} P(t,x,y)$. A PDF $\pi : \mathbb{R} \to [0,\infty)$ is called a stationary PDF or an invariant PDF for X if

$$\pi(y) = \int_{-\infty}^{\infty} p(t,x,y)\pi(x)\,dx \quad \text{for } y \in \mathbb{R}.$$

This means that if $X(s)$ has PDF π for an $s \ge 0$, then also $X(t+s)$ has PDF π for $t > 0$. Making use of the formula

$$f_{X(t_1),\ldots,X(t_n)}(x_1,\ldots,x_n) = f_{X(t_1)}(x_1)\prod_{i=2}^{n} p(t_i - t_{i-1}, x_{i-1}, x_i) \quad \text{for } x_1,\ldots,x_n \in \mathbb{R},$$

for the joint PDF of $(X(t_1),\ldots,X(t_n))$ for $0 \le t_1 < \cdots < t_n$, it follows that if X_0 has a stationary PDF π then X is a stationary process, which is to say that

Proved in exercises.

$$f_{X(t_1+h),\ldots,X(t_n+h)}(x_1,\ldots,x_n) = f_{X(t_1+h),\ldots,X(t_n+h)}(x_1,\ldots,x_n) \quad \text{for } h > 0.$$

By applying the Kolmogorov forward equation to the definition of a stationary PDF we obtain an ODE that π must satisfy:

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty}\left[\left(\frac{1}{2}\frac{\partial^2}{\partial y^2}\sigma(y)^2 - \frac{\partial}{\partial y}\mu(y) - \frac{\partial}{\partial t}\right)p(t,x,y)\right]\pi(x)\,dx \\ &= \left(\frac{1}{2}\frac{\partial^2}{\partial y^2}\sigma(y)^2 - \frac{\partial}{\partial y}\mu(y)\right)\pi(y) \\ &= (L^*\pi)(y). \end{aligned}$$

$\Rightarrow \frac{1}{2}\frac{\partial}{\partial y}(\sigma(y^2)\pi(y)) - \mu(y)\pi(y) = C$

Formally a solution to this ODE is given by

$$\pi(x) = \frac{C}{\sigma(x)^2}\exp\left(\int_{x_0}^{x}\frac{2\mu(y)}{\sigma(y)^2}\,dy\right) \quad \text{for } x \in \mathbb{R}.$$

Here $C > 0$ and $x_0 \in \mathbb{R}$ are constants that are determined by the normalisation $\int_{-\infty}^{\infty}\pi(x)\,dx = 1$. [We really have only one "free" constant in the above solution to the second order ODE before normalisation (as C and x_0 "interact"): The reason is that the second free constant in the general solution to the ODE disappears to even make normalisation possible.]

Under technical conditions the SDE has the stationary PDF of the previous paragraph provided that the normalisation is possible and that the following two integrals both are infinite:

$$\int_{-\infty}^{x_0}\exp\left(-\int_{x_0}^{x}\frac{2\mu(y)}{\sigma(y)^2}\,dy\right)dx \quad \text{and} \quad \int_{x_0}^{\infty}\exp\left(-\int_{x_0}^{x}\frac{2\mu(y)}{\sigma(y)^2}\,dy\right)dx.$$

This of course is the necessary and sufficient criteria for recurrence of a diffusion we have seen earlier.

Example $dX(t) = -\alpha X(t)dt + \sigma dB(t)$ is reccurent for $\alpha \ge 0$ according to previous example and $\pi(x) = \frac{c}{\sigma^2}\exp\left(-\frac{\alpha x^2}{\sigma^2}\right)$ is a normal stationary with suitable c for $\alpha > 0$.

If there exists a (non probability) measure ν on $(\mathbb{R},\mathcal{B})$ with $\nu(\mathbb{R}) = \infty$ such that

$$\nu(B) = \int_{-\infty}^{\infty}\left(\int_{y\in B} dP(t,x,y)\right)d\nu(x) = \int_{-\infty}^{\infty}P(t,x,B)\,d\nu(x) \quad \text{for } B \in \mathcal{B},$$

then ν is called an invariant measure for X. For example, while BM does not have a stationary CDF or PDF it has the usual Euclidian measure of length $d\nu(x) = dx$ as invariant measure. This follows from the fact that $p(t,x,y)$ for BM is a PDF both viewed as a function of y and as a function of x.