Numerical Lecture Notes

Thm Let X and $\hat{X}$ be solutions to SDE with initial values X_0 and $\hat{X}_0$. Then for $C = C(L,T)$

$$E\left(\max_{0\le t\le T} |\hat{X}(t) - X(t)|^2\right) \le C\, E(|\hat{X}_0 - X_0|^2).$$

Proof $\phi(t) = E\left(\max_{0\le s\le t} |\hat{X}(s) - X(s)|^2\right)$

$$= E\left(\max_{0\le s\le t}\left|\hat{X}_0 - X_0 + \int_0^s [\mu(\hat{X}(r),r) - \mu(X(r),r)]\,dr + \int_0^s [\sigma(\hat{X}(r),r) - \sigma(X(r),r)]\,dB(r)\right|^2\right)$$

$[(x+y+z)^2 \le 3x^2+3y^2+3z^2]$ $\le 3E(|\hat{X}_0 - X_0|^2) + 3E\left(\max_{0\le s\le t}\left|\int_0^s (\hat{\mu}-\mu)\,dr\right|^2\right) + 3E\left(\max_{0\le s\le t}\left|\int_0^s (\hat{\sigma}-\sigma)\,dB\right|^2\right)$

[Cauchy-Schwarz, Doob] $\le 3E(|\hat{X}_0 - X_0|^2) + 3E\left(\max_{0\le s\le t} \int_0^s 1^2\,dr \int_0^s (\hat{\mu}-\mu)^2\,dr\right) + 12\,E\left(\left|\int_0^t (\hat{\sigma}-\sigma)\,dB\right|^2\right)$

[isometry] $\le 3E(|\hat{X}_0 - X_0|^2) + 3E\left(t\int_0^t |\hat{\mu}-\mu|^2\,dr\right) + 12\,E\left(\int_0^t (\hat{\sigma}-\sigma)^2\,dr\right)$

[Lipschitz] $\le 3E(|\hat{X}_0 - X_0|^2) + 3TL^2 \int_0^t E((\hat{X}(r)-X(r))^2)\,dr + 12L^2\int_0^t E((\hat{X}(r)-X(r))^2)\,dr$

$\le 3E(|\hat{X}_0 - X_0|^2) + (3TL^2 + 12L^2)\int_0^t \phi(r)\,dr \overset{\text{Grönwall}}{\Longrightarrow}$

$$\phi(t) \le 3E(|\hat{X}_0 - X_0|^2)\, e^{(3TL^2+12L^2)t} \quad \text{for } t\in[0,T]$$

finish by taking $t=T$. #

Thm For $C = C(L,T)$ we have

$$E(|X(t) - X(s)|^2) \le C\,(1 + E(X_0^2))\,|t-s| \quad \text{for } s,t\in[0,T].$$

Proof $E(|X(t)-X(s)|^2) = E\left(\left|\int_s^t \mu(X(r),r)\,dr + \int_s^t \sigma(X(r),r)\,dB(r)\right|^2\right)$

$\left[\substack{(x+y)^2\\ \le 2x^2+2y^2}\right] \le 2E\left(\left(\int_s^t \mu\,dr\right)^2\right) + 2E\left(\left(\int_s^t \sigma\,dB\right)^2\right)$

$\left[\substack{\text{Cauchy Schwarz}\\ \text{isometry}}\right] \le 2\int_s^t 1^2\,dr \int_s^t E(\mu^2)\,dr + 2E\left(\int_s^t \sigma^2\,dr\right)$

$[\text{linear growth}] \le 4(t-s)\int_s^t L^2 E(1^2+|X(r)|^2)\,dr + 4\int_s^t L^2 E(1^2+|X(r)|^2)\,dr$

$\left[\substack{\text{Theorem 1}\\ \text{Stig's notes}}\right] \le 4(T+1)L^2\left(1+C(1+E(X_0^2))\right)(t-s)$ #

Notation $0=t_0<t_1<\ldots<t_N=T$, $h_n = t_{n+1}-t_n$, $h=\max_n h_n \le 1$, $\Delta B_n = B(t_{n+1})-B(t_n)$

$$X(t_{n+1}) = X(t_n) + \int_{t_n}^{t_{n+1}} \mu(X(s),s)\,ds + \int_{t_n}^{t_{n+1}} \sigma(X(s),s)\,dB(s), \quad X(0)=X_0$$

$$\begin{aligned} Y_{n+1} &= Y_n + \mu(Y_n,t_n)\int_{t_n}^{t_{n+1}} ds + \sigma(Y_n,t_n)\int_{t_n}^{t_{n+1}} dB(s) \\ &= Y_n + \mu(Y_n,t_n)h_n + \sigma(Y_n,t_n)\Delta B_n, \quad Y_0 \approx X_0 \end{aligned}$$

$Y(t) = Y_0 + \int_0^t \bar{\mu}(s)\,ds + \int_0^t \bar{\sigma}(s)\,dB(s)$ satisfies $Y(t_n) = Y_n$ for

$\bar{\mu}(s) = \mu(Y_n,t_n)$ and $\bar{\sigma}(s) = \sigma(Y_n,t_n)$ for $s\in[t_n,t_{n+1}]$.

Thm If $E(|Y_0-X_0|^2) \le kh$ and $E(X_0^2)\le M$ then

$$E\left(\max_{0\le t\le T}|Y(t)-X(t)|^2\right) \le Ch$$

Proof $\phi(t) = E\left(\max_{0\le s\le t} |Y(s) - X(s)|^2\right)$

$$\begin{bmatrix}(x+y+z)^2 \\ \le 3(x^2+y^2+z^2)\end{bmatrix} \le 3E\left((Y_0 - X_0)^2\right) + 3E\left(\max_{0\le s\le t}\left(\int_0^s (\bar{\mu}(r) - \mu(r))\,dr\right)^2\right) + 3E\left(\max_{0\le s\le t}\left|\int_0^s (\bar{\sigma}(r) - \sigma(r))\,dB(r)\right|^2\right)$$

$$\begin{bmatrix}\text{Cauchy Schwarz} \\ \text{Doob}\end{bmatrix} \le 3Kh + 3\int_0^t 1^2\,dr \int_0^t (\bar{\mu}(r) - \mu(r))^2\,dr + 12E\left(\left(\int_0^t (\bar{\sigma}(r) - \sigma(r))\,dB(r)\right)^2\right)$$

$$[\text{isometry}] \le 3Kh + 3T\int_0^t E\left((\bar{\mu}(r) - \mu(r))^2\right)dr + 12\int_0^t E\left((\bar{\sigma}(r) - \sigma(r))^2\right)dr = (*)$$

$$E\left(|\bar{\mu}(s) - \mu(s)|^2\right) \le 2E\left(|\mu(Y(t_n), t_n) - \mu(X(s), t_n)|^2\right) + 2E\left(|\mu(X(s), t_n) - \mu(X(s), s)|^2\right)$$

$$\le 2L^2 E\left(|Y(t_n) - X(s)|^2\right) + 2L^2|t_n - s|^2$$

$$\le 4L^2 E\left(|Y(t_n) - X(t_n)|^2\right) + 4L^2 E\left(|X(t_n) - X(s)|^2\right) + 2L^2 h_n^2$$

$$\le 4L^2 E\left(\max_{0\le r\le s}|Y(s) - X(s)|^2\right) + \underbrace{4L^2 C|t_n - s| + 2L^2 h_n^2}_{\le Ch_n \,\le\, Ch} \quad \text{for } s \in [t_n, t_{n+1})$$

We get same with other constants for $E\left((\bar{\sigma}(s) - \sigma(s))^2\right)$ and so we have

$\phi(t) = Ch + \int_0^t \phi(s)\,ds$ for $t \in [0, T]$ so that

$\phi(t) = Ch\,e^{Bt}$ for $t \in [0, T]$. #

Itô-Taylor expansion

$$dX(t) = \mu(X(t),t)\,dt + \sigma(X(t),t)\,dB(t)$$

$$X(t) = X(t_0) + \int_{t_0}^{t} \mu(X(s),s)\,ds + \int_{t_0}^{t} \sigma(X(s),s)\,dB(s) \qquad (*)$$

$$f(X(s),s) = f(X(t_0),t_0) + \int_{t_0}^{s} f'_x(X(r),r)\left[\mu(X(r),r)\,dr + \sigma(X(r),r)\,dB(r)\right]$$
$$+ \int_{t_0}^{s} \tfrac{1}{2} f''_{xx}(X(r),r)\,\sigma(X(r),r)^2\,dr + \int_{t_0}^{s} f'_t(X(r),r)\,dr \qquad (**)$$

Using $(**)$ with $f(x,t)=\mu(x,t)$ and $f(x,t)=\sigma(x,t)$ in $(*)$ gives

$$\underline{X(t)} = \underline{X(t_0)} + \underline{(t-t_0)\,\mu(X(t_0),t_0)} + \int_{t_0}^{t}\int_{t_0}^{s} \mu'_x(X(r),r)\,\mu(X(r),r)\,dr\,ds$$
$$+ \int_{t_0}^{t}\int_{t_0}^{s} \mu'_x(X(r),r)\,\sigma(X(r),r)\,dB(r)\,ds + \int_{t_0}^{t}\int_{t_0}^{s} \tfrac{1}{2}\mu''_{xx}(X(r),r)\,\sigma(X(r),r)^2\,dr\,ds$$
$$+ \int_{t_0}^{t}\int_{t_0}^{s} \mu'_t(X(r),r)\,dr\,ds + \underline{(B(t)-B(t_0))\,\sigma(X(t_0),t_0)}$$
$$+ \int_{t_0}^{t}\int_{t_0}^{s} \sigma'_x(X(r),r)\,\mu(X(r),r)\,dr\,dB(s) + \int_{t_0}^{t}\int_{t_0}^{s} \sigma'_x(X(r),r)\,\sigma(X(r),r)\,dB(r)\,dB(s)$$
$$+ \int_{t_0}^{t}\int_{t_0}^{s} \tfrac{1}{2}\sigma''_{xx}(X(r),r)\,\sigma(X(r),r)^2\,dr\,dB(s) + \int_{t_0}^{t}\int_{t_0}^{s} \sigma'_t(X(r),r)\,dr\,dB(s)$$

The underlined parts of above make up the motivation for the Euler (Maruyama) method for numerical approximative solution of the SDE. By adding the part similarly approximated by

$$(\sigma'_x\sigma)(X(t_0),t_0)\int_{t_0}^{t}\int_{t_0}^{s} dB(r)\,dB(s) = (\sigma'_x\sigma)(X(t_0),t_0)\int_{t_0}^{t}(B(s)-B(t_0))\,dB(s)$$

$$= (\sigma'_x\sigma)(X(t_0),t_0)\left(\tfrac{1}{2}\left(B(t)^2-B(t_0)^2-(t-t_0)\right) - B(t_0)(B(t)-B(t_0))\right) = (\sigma'_x\sigma)(X(t_0),t_0)\left(\tfrac{1}{2}(B(t)-B(t_0))^2 - \tfrac{1}{2}(t-t_0)\right)$$

we get the most used higher order Milstein method.

Weak numerical solutions

In a strong solution one is given $\mu(x,t), \sigma(x,t)$ and $B(t)$ and should use them to manufacture $X(t)$.

In a weak solution one is given $\mu(x,t), \sigma(x,t)$ and should use them to manufacture $B(t)$ and $X(t)$.

In theory the difference is bigger than it may seem in practice. There exist SDE which has unique weak solution but no strong solution, e.g., Tanaka SDE

$$dX(t) = \text{sign}(X(t))\, dB(t).$$

Uniqueness of strong solution means
$P(X_1(t) = X_2(t)) = 1$ for all solutions $X_1(t)$ and $X_2(t)$

Uniqueness of weak solution means

$$F_{X_1(t_1),\dots,X_n(t_n)}(x_1,\dots,x_n) = F_{X_2(t_1),\dots,X_2(t_n)}(x_1,\dots,x_n).$$

The numerical methods we have seen so far are strong numerical methods in the sense that if we are given $\mu(x,t)$, $\sigma(x,t)$ and $B(t)$ we can plug in them in method and get $X(t)$ – example Euler, implicit Euler, Milstein.

When we use this methods in computer no one gives us $B(t)$ so we still create $B(t)$ ourselves to get going.

Example (Langevin equation) $dX(t) = -\alpha X(t)\,dt + \sigma\,dB(t)$

Euler method $Y_{n+1} = Y_n - \alpha Y_n (t_{n+1} - t_n) + \sigma\left(B(t_{n+1}) - B(t_n)\right)$

is used in applied lecture. Exact solution is

$$X(t) = e^{-\alpha t}\left(X_0 + \int_0^t \sigma e^{\alpha r}\,dB(r)\right)$$

which gives

$$X(t) = e^{-\alpha(t-s)} X(s) + \underbrace{e^{-\alpha t}\int_s^t \sigma e^{\alpha r}\,dB(r)}_{N\left(0,\ \frac{\sigma^2}{2\alpha}\left(1-e^{-2\alpha(t-s)}\right)\right)}$$

$$f_{X(t)|X(s)}(y|x) = \frac{1}{\sqrt{2\pi\,\frac{\sigma^2}{2\alpha}\left(1-e^{-2\alpha(t-s)}\right)}}\, e^{-\frac{y-e^{-\alpha(t-s)}x}{\frac{\sigma^2}{\alpha}\left(1-e^{-2\alpha(t-s)}\right)}}$$

so we can create an exact weak solution at times $0 = t_0 < t_1 < \ldots < t_N = T$ by doing

$$X(t_{n+1}) = e^{-\alpha(t_{n+1}-t_n)} X(t_n) + NID\left(0, \frac{\sigma^2}{2\alpha}\left(1-e^{-2\alpha(t_{n+1}-t_n)}\right)\right)$$

$n = 0, \ldots, N-1$, $X(0) = x_0$

We cannot do an exact theoretical solution in reality because we cannot calculate

$\int_0^t e^{\alpha r} dB(r)$ exactly!

Remember that solution to SDE are Markov processes. Their transition probability densities

$$p(y,t,x,s) = f_{X(t)|X(s)}(y|x)$$

satisfy the Kolmogorov backward PDE

$$\mu(x,s)\frac{\partial p}{\partial x} + \frac{\sigma(x,s)^2}{2}\frac{\partial^2 p}{\partial x^2} + \frac{\partial p}{\partial s} = 0$$

and by solving that PDE for a $p(y,t,x,s)$ that satisfies

$$\lim_{s \uparrow t} \int_{-\infty}^{+\infty} g(y)\, p(y,t,x,s)\, dy = g(x)$$

for smooth $g(x)$ one can find $p(y,t,x,s)$.

But given an OK family of transition PDF there exists a Markov process that has these transition PDF's and that Markov process is weak solution to SDE.

One can create an exact weak solution by the recursive scheme

simulate $X(t_{n+1})$ with PDF $p(\cdot, t_{n+1}, X(t_n), t_n)$
for $n = 0, \dots, N-1$,

$$X(0) = x_0$$

To create a exact weak solution is in principle same as finding the Markov transition density.

And a numerical weak scheme same as approximating the Markov transition density.

The Euler method

$$Y_{n+1} = Y_n + \mu(Y_n, t_n)(t_{n+1} - t_n) + \sigma(Y, t_n)(B(t_{n+1}) - B(t_n))$$

we have seen has strong convergence

$$\sqrt{E\left(\max_{0 \le n \le N} |Y_n - X(t_n)|^2\right)} \le C h^{1/2} \text{ for } h = \max_{0 \le n \le N-1} t_{n+1} - t_n$$

so

$$E\left(\max_{0 \le n \le N} |Y_n - X(t_n)|\right) \le \sqrt{E\left(\max_{0 \le n \le N} |Y_n - X(t_n)|^2\right)} \le C h^{1/2}$$

But if we use same Euler method to just approximate $g(X(T))$ with $g(Y_N)$

we have $|E(g(Y_N)) - E(g(X(T)))| \le Ch$.

The proof of this uses some quite difficult PDE techniques.

If we want to find $E(g(\underline{X}(T)))$ in reality we have to generate many independent observations $\{g(\underline{Y}_N^{(i)})\}_{i=1}^{n}$ of $g(\underline{Y}_N)$ and then use Monte Carlo

$$E(g(\underline{X}(T))) \approx E(g(\underline{Y}_N)) \approx \frac{1}{n}\sum_{i=1}^{n} g(\underline{Y}_N^{(i)}).$$