TMS165/MSA350 Stochastic Calculus
Written home exam Tuesday 27 October 2020 8.30–12.30

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AIDS: All aids are permitted. (See the Canvas course "Ordinarie tentamen Modul: 0104, TMS165 / MSA350" with instructions for this exam for clarifications.)

GRADES: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Througout this exam $B = \{B(t)\}_{t\geq 0}$ denotes a Brownian motion.

Task 1. Let X be a standard normal $\mathrm{N}(0,1)$ random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Calculate $\mathbf{E}\{X|\mathcal{G}\}$ for $\mathcal{G} = \{\emptyset, \{X<0\}, \{X\geq 0\}, \Omega\}$. **(5 points)**

Task 2. Find the quadratic variation process for the process $X(t) = \exp\{\mathrm{e}^{B(t)}\}$. Is $X(t)$ an Itô process? **(5 points)**

Task 3. Why do most SDE we talk about have a constant initial value? **(5 points)**

Task 4. Find the solution to the PDE

$$\frac{1+x^2}{2}\frac{\partial^2 f(x,t)}{\partial x^2} + \big(\sqrt{1+x^2} + \frac{x}{2}\big)\frac{\partial f(x,t)}{\partial x} + \frac{\partial f(x,t)}{\partial t} = 0 \quad \text{for } t\in[0,T)$$

with terminal value $f(x,T) = \sinh^{-1}(x)$. **(5 points)**

Task 5. Is it possible to make the stochastic exponential $\mathcal{E}(B(t))$ of Brownian motion a Brownian motion by means of change of measure? **(5 points)**

Task 6. By means of so called Itô-Taylor expansion one can derive higher order methods than the ordinary Euler scheme for numerical solution of SDE that potentially have better convergence speed than the Euler scheme. Explain what problem can occur when one tries to implement such higher order methods. **(5 points)**

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Solutions to Written Exam 27 October 2020

Task 1. As $\mathbf{E}\{X^+\} = \frac{1}{\sqrt{2\pi}}$ we have $\mathbf{E}\{X|\mathcal{G}\}(\omega) = \frac{\overset{2}{\cancel{1}}}{\sqrt{2\pi}} \mathbf{1}_{\{X \geq 0\}}(\omega) - \frac{\overset{2}{\cancel{1}}}{\sqrt{2\pi}} \mathbf{1}_{\{X<0\}}(\omega)$.

Task 2. As $d(\exp\{e^{B(t)}\}) = e^{B(t)} \exp\{e^{B(t)}\}\, dB(t) + \frac{1}{2}(e^{B(t)} + e^{2B(t)}) \exp\{e^{B(t)}\}\, dt$ we see that $X(t)$ is an Itô process with quadratic variation process $\int_0^t e^{2B(s)} \exp\{2e^{B(s)}\}\, ds$.

Task 3. Because the solution to the SDE is adapted to $\{\mathcal{F}_t^B\}_{t\geq 0}$ where $\mathcal{F}_0^B = \{\emptyset, \Omega\}$ since $B(0)$ is a constant and the only $\{\emptyset, \Omega\}$-measurable random variables are constants.

Task 4. According to Theorem 6.6 in Klebaner's book the solution is $f(x,t) = \mathbf{E}\{\sinh^{-1}(X(T))|X(t) = x\}$ where by the solved exercises $X(t) = \sinh(t + B(t))$ which readily gives $f(x,t) = T + \sinh^{-1}(x) - t$.

Task 5. No because for that it is required that the quadratic variation process is the same as for BM $[B(t)] = t$ while $[\mathcal{E}(B(t))] = \int_0^t \mathcal{E}(B(s))^2\, ds = \int_0^t e^{2B(s)-s}\, ds$.

Task 6. One encounters multiple Itô integrals that cannot be calculated exactly.