TMS165/MSA350 Stochastic Calculus
Written home exam Monday 4 January 2021 2–6 PM

TEACHER: Patrik Albin 031 7723512 palbin@chalmers.se .

AIDS: All aids are permitted. (See the Canvas course "Omtentamen 1 Modul: 0104, TMS165 / MSA350" with instructions for this exam for clarifications.)

GRADES: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Througout this exam $B = \{B(t)\}_{t\geq 0}$ denotes a Brownian motion.

Task 1. Find $\mathbf{E}\{X|\sigma(Z)\}$ when X and Y are independent unit mean exponentially distributed random variables and $Z = X + Y$. **(5 points)**

Task 2. Consider a stochastic integral process $\{\int_0^t X\,\Delta B\}_{t=0}^{\infty}$ defined by $\int_0^t X\,\Delta B = \lim_{\max_{1\leq i\leq n} t_i^n - t_{i-1}^n \downarrow 0} \sum_{i=1}^n X(t_i^n)\,(B(t_i^n) - B(t_{i-1}^n))$ for grids $0 = t_0^n < t_1^n < \ldots < t_n^n = t$ and continuous adapted processes $\{X(t)\}_{t\geq 0}$. Express $\int_0^t B^2\,\Delta B$ as an Itô process.

(5 points)

Task 3. Does the Itô process $X(t) = B(t)^6$ satisfy any diffusion type SDE?

(5 points)

Task 4. For which choices of the parameter $\alpha > 0$ does the SDE $dX(t) = X(t)^{\alpha}\,dt + X(t)\,dB(t)$, $X(0) = 1$, have a stationary distribution? **(5 points)**

Task 5. Let $X(\omega) = \omega$ for $\omega \in \Omega = \mathbb{R}$ and $\mathbf{P}(A) = \int_A \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}\,dx$ for (Borel sets) $A \subseteq \mathbb{R}$. What is the probability distribution of X under $\mathbf{P}$? Find two different equivalent to $\mathbf{P}$ probability measures $\mathbf{Q}_1$ and $\mathbf{Q}_2$ on the (Borel) subsets of $\mathbb{R}$ such that $\mathbf{E}_{\mathbf{Q}_1}\{X\} = \mathbf{E}_{\mathbf{Q}_2}\{X\} = 1$ and $\mathbf{E}_{\mathbf{Q}_1}\{X^2\} = \mathbf{E}_{\mathbf{Q}_2}\{X^2\} = 2$. **(5 points)**

Task 6. Which SDE does the iterative scheme $Y_{i+1} = Y_i + Y_{i+1}(t_{i+1}^n - t_i^n) + Y_{i+1} \times (B(t_{i+1}^n) - B(t_i^n))$ for $i = 0, \ldots, n-1$, $Y_0 = 1$, become a solution of as $0 = t_0^n < t_1^n < \ldots < t_n^n = t$ satisfies $\max_{1\leq i\leq n} t_i^n - t_{i-1}^n \downarrow 0$? How can one be sure the scheme converges?

(5 points)

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Solutions to Written Exam 4 January 2021

Task 1. As $f_{X,Z}(x,z) = \mathrm{e}^{-z}$ for $0 \leq x \leq z$ and $f_Z(z) = z\,\mathrm{e}^{-z}$ for $z \geq 0$ we have $\mathbf{E}\{X|Z=z\} = \int_{-\infty}^{\infty} x\, f_{X,Z}(x,z)/f_Z(z)\, dx = \int_0^z x/z\, dx = z/2$ giving $\mathbf{E}\{X|\sigma(Z)\} = Z/2$.

Task 2. $\int_0^t B^2\,\Delta B \leftarrow \sum_{i=1}^n B(t_i^n)^2\,(B(t_i^n) - B(t_{i-1}^n)) = \sum_{i=1}^n (B(t_i^n) - B(t_{i-1}^n))^3 + 2\sum_{i=1}^n B(t_{i-1}^n)\,(B(t_i^n) - B(t_{i-1}^n))^2 + \sum_{i=1}^n B(t_{i-1}^n)^2\,(B(t_i^n) - B(t_{i-1}^n)) \to 0 + 2\int_0^t B(s)\,ds + \int_0^t B^2\,dB$.

Task 3. As $dX(t) = 6\,B(t)^5\,dB(t) + 15\,B(t)^4\,dt = 6\,B(t)^5\,dB(t) + 15\,X(t)^{2/3}\,dt$ where $B(t)^5$ is not a function of $X(t)$ the answer is negative: $X(t)$ doesn't satisfy a diffusion type SDE.

Task 4. We can choose a constant $C > 0$ such that

$$\pi(x) = \frac{C}{\sigma(x)^2}\exp\bigg\{\int_1^x \frac{2\mu(y)\,dy}{\sigma(y)^2}\bigg\} = \begin{cases} C\,x^{-2}\,\exp\{2(x^{\alpha-1}-1)/(\alpha-1)\} & \text{for } \alpha \neq 1 \\ C & \text{for } \alpha = 1 \end{cases}$$

satisfies $\int_0^\infty \pi(x)\,dx = 1$ if and only if $\alpha \in (0,1)$. In that case

$$\int_0^1 \exp\bigg\{-\int_1^x \frac{2\mu(y)\,dy}{\sigma(y)^2}\bigg\}dx = \int_1^\infty \exp\bigg\{-\int_1^x \frac{2\mu(y)\,dy}{\sigma(y)^2}\bigg\}dx = \infty$$

so that $\pi(x)$ is the stationary PDF.

Task 5. As shown in an exercise of the course X is $\mathrm{N}(0,1)$. Further $\mathbf{E}\{X\} = 1$ and $\mathbf{E}\{X^2\} = 2$ under, e.g., $\mathbf{Q}_1(A) = \int_A \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-(x-1)^2/2}\,dx$ and $\mathbf{Q}_2(A) = \int_A \frac{1}{\sqrt{2}}\,\mathrm{e}^{-\sqrt{2}\,|x-1|}\,dx$.

Task 6. The SDE $dX(t) = 2\,X(t)\,dt + X(t)\,dB(t)$, $X(0) = 1$, has $[X(t),t] = 0$ and $[X(t),B(t)] = \int_0^t X(s)\,ds$, so that the iterative scheme solves this SDE. The coefficients of the SDE are globally Lipschitz in the space variable uniformly in time ensuring convergence of the Euler scheme $Y_{i+1} = Y_i + 2\,Y_i(t_{i+1}^n - t_i^n) + Y_i(B(t_{i+1}^n) - B(t_i^n))$ to $X(t)$ as $\max_{1\leq i\leq n} t_i^n - t_{i-1}^n \downarrow 0$. But then the difference between the schemes

$$[Y_i + Y_{i+1}(t_{i+1}^n - t_i^n) + Y_{i+1}(B(t_{i+1}^n) - B(t_i^n))] - [Y_i + 2Y_i(t_{i+1}^n - t_i^n) + Y_i(B(t_{i+1}^n) - B(t_i^n))]$$
$$\to [X(t),t] - \int_0^t X(s)\,ds + [X(t),B(t)] = 0.$$