TMS165/MSA350 Stochastic Calculus
Written home exam Friday 20 August 2021 8.30–12.30

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AIDS: All aids are permitted. (See the Canvas course "Omtentamen 2 Modul: 0104, TMS165 / MSA350" with instructions for this exam for clarifications.)

GRADES: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Througout this exam $B = \{B(t)\}_{t\geq 0}$ denotes a Brownian motion.

Task 1. Find $\mathbf{E}\{X|\mathcal{G}\}$ when X is a standard normal distributed random variable and $\mathcal{G} = \big\{\emptyset, \{X \geq 0\}, \{X < 0\}, \Omega\big\}$. **(5 points)**

Task 2. Is $\{\int_0^t \mathrm{e}^{B(s)^2}\, dB(s)\}_{t\geq 0}$ a martingale? **(5 points)**

Task 3. Find the quadratic variation process of the Itô process $\{B(t)\int_0^t B(s)\, dB(s)\}_{t\geq 0}$. **(5 points)**

Task 4. Solve the SDE $dY(t) = 3\,Y(t)^{1/3}\,dt + 3\,Y(t)^{2/3}\,dB(t)$ for $t \geq 0$ where $Y(t)^{\rho} = |Y(t)|^{\rho}\,\mathrm{sign}(Y(t))$. **(5 points)**

Task 5. Can a standard normal random variable be made a unit mean Poisson distributed random variable by means of change of probability measure? **(5 points)**

Task 6. Which SDE is the scheme $X_{n+1} = X_n - X_{n+1}(t_{n+1} - t_n) + (B(t_{n+1}) - B(t_n))$ for $n = 0, \ldots, N-1$, $0 = t_0 < t_1 < \ldots < t_N = T$, an approximative numerical solution of? **(5 points)**

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Task 1. $\mathbf{E}\{X|\mathcal{G}\} = 2\int_0^\infty x\,\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-x^2/2}\,dx$ for $\omega\in\{X\geq 0\}$ and $\mathbf{E}\{X|\mathcal{G}\} = 2\int_{-\infty}^0 x\,\frac{1}{\sqrt{2\pi}}$ $\times\,\mathrm{e}^{-x^2/2}\,dx$ for $\omega\in\{X<0\}$ as that is easily seen to fit with the defining requirement that $\mathbf{E}\{X|\mathcal{G}\}$ shall be $\mathcal{G}$-measurable with $\mathbf{E}\big\{\mathbf{E}\{X|\mathcal{G}\}\mathbf{1}_B\big\} = \mathbf{E}\{X\mathbf{1}_B\}$ for $B\in\mathcal{G}$.

Task 2. As $\{\mathrm{e}^{B(t)^2}\}_{t\in[0,T]}$ is not in E_T for T large enough this question cannot be answered by what we have learnt.

Task 3. $\int_0^t\big((\int_0^s B\,dB)\,dB(s)+B(s)^2\,dB(s)+B(s)\,dB(s)^2\big)^2 = \int_0^t\big((\int_0^s B\,dB)^2+2\,(\int_0^s B$ $dB)B(s)^2+B(s)^4\big)ds$.

Task 4. $X(t)=Y(t)^{1/3}$ gives $dX(t)=\ldots=dB(t)$ so that $X(t)=B(t)+X(0)$ and $Y(t)=(B(t)+Y(0)^{1/3})^3$.

Task 5. No because a normal probability measure is singular wrt. a Poisson probability measure.

Task 6. Since $[X(t),t](t)=0$ the scheme solves the same SDE as the scheme $X_{n+1}=X_n-X_n(t_{n+1}-t_n)+(B(t_{n+1})-B(t_n))$ which in turn is the Euler scheme for the Langevin SDE $dX(t)=-X(t)\,dt+dB(t)$ for $t\in[0,T]$.