Stochastic Calculus Part II Fall 2008

Hand in 1

1. A process $\{X_t\}_{t\geq 0}$ is called *weakly stationary* if

1. $m_t := \mathrm{E}[X_t] = m \in \mathbb{R}, \forall t \geq 0$ (are all well defined)
2. Let $h \in \mathbb{R}$ be fixed. $\gamma(X_t, X_{t+h}) := \mathrm{Cov}(X_t, X_{t+h}) = \mathrm{Cov}(X_h, X_0)$ for all $t \geq 0$ (where $t+h \geq 0$ with the covariances being well defined)

A process $\{X_t\}_{t\geq 0}$ is called *stationary* if
for $n \in \mathbb{N}$, $t_1, \ldots, t_n \geq 0$, and $h \in \mathbb{R}$ where $t_1 + h, \ldots, t_n + h \geq 0$
$(X_{t_1}, \ldots, X_{t_n}) \stackrel{d}{=} (X_{t_1+h}, \ldots, X_{t_n+h})$

Show that
i) If X_t is a stationary process, then it is also weakly stationary (assuming that all moments involved are well defined).
ii) Assume that X_t is a Gaussian process. Show that in this case weak stationarity implies stationarity (i.e. in this case the two types of stationarity are equivalent).

2. Let X_t be a diffusion with coefficients $\mu(x) = cx$ and $\sigma(x) = 1$.
a) Give its generator.
b) Show that the process $X_t^2 - 2c \int_0^t X_s^2\, ds - t$ is a martingale.

3. Is the *Ornstein-Uhlenbeck* (OU) process (given in example 5.6, p.127) recurrent/transient?
Show that the limiting distribution of the OU process exists and that this distribution is given by its stationary distribution. (X_0 is treated as non-random)
(Hint: What does chapter 4 say about the distribution of $\sigma e^{-\alpha t} \int_0^t e^{\alpha s}\, dB_s$? If you know this distribution you can take the limit of it (the distribution) to get the required limit distribution. These two distributions will belong to the same family of distributions.)