TMS170/MSA360 Stochastic Calculus Part II, Fall 2008

Solutions to Exercise Session 1

November 6, 2008

Exercise 6.5

For the diffusion process $X(t)$ that solves the time homogeneous SDE

$$dX(t) = \mu(X(t))\,dt + \sigma(X(t))\,dB(t)$$

with coefficients $\mu(x) = 2x$ and $\sigma^2(x) = 4x$, the generator (6.30) takes the form

$$L = \frac{1}{2}\sigma^2(x)\,\frac{d^2}{dx^2} + \mu(x)\,\frac{d}{dx} = 2x\,\frac{d^2}{dx^2} + 2x\,\frac{d}{dx}\,.$$

The general solution to the equation (6.48) $Lf = 0$ is given by (6.50) as

$$f(x) = \int_{x_0}^{x} \exp\biggl\{-\int_{y_0}^{y} \frac{2\mu(u)}{\sigma^2(u)}\,du\biggr\}\,dy = \int_{x_0}^{x} \mathrm{e}^{y_0 - y}\,dy = \mathrm{e}^{y_0}\bigl(\mathrm{e}^{-x_0} - \mathrm{e}^{-x}\bigr) = C_1\,\mathrm{e}^{-x} + C_2,$$

where $x_0, y_0, C_1, C_2 \in \mathbb{R}$ are constants. A convenient martingale M_f associated with $X(t)$ is thus given by equation (6.37) (with this particular choice of the function f):

$$M_f(t) = f(X(t)) - \int_0^t Lf((X(s))\,ds = f(X(t)) = C_1\,\mathrm{e}^{-X(t)} + C_2.$$

Of course, the fact that $f(X(t))$ is a martingale can also be checked by means of using the Itô formula to establish that

$$df(X(t)) = -2\,f(X(t))\sqrt{X(t)}\,dB(t) \quad\Rightarrow\quad f(X(t)) = -2\int_0^t f(X(s))\sqrt{X(s)}\,dB(s).$$

(The reader interested in this calculation has to fill in a few details herself/himself!)
For the process $Y(t) = \sqrt{X(t)}$ the Itô formula gives

$$\begin{aligned} dY(t) &= \frac{1}{2\sqrt{X(t)}}\,dX(t) - \frac{1}{2}\,\frac{1}{4\,X(t)^{3/2}}\,d[X,X](t) \\ &= \frac{2X(t)\,dt + 2\sqrt{X(t)}\,dB(t)}{2\sqrt{X(t)}} - \frac{4X(t)\,dt}{8\,X(t)^{3/2}} \\ &= \left(Y(t) - \frac{1}{2\,Y(t)}\right)dt + dB(t)\,. \end{aligned}$$

Hence Y is a time homogeneous diffusion process

$$dY(t) = \mu(Y(t))\,dt + \sigma(Y(t))\,dB(t)$$

with coefficients $\mu(x) = x - 1/(2x)$ and $\sigma(x) = 1$. The generator of this SDE is

$$L = \frac{1}{2}\sigma^2(x)\,\frac{d^2}{dx^2} + \mu(x)\,\frac{d}{dx} = \frac{1}{2}\,\frac{d^2}{dx^2} + \left(x - \frac{1}{2x}\right)\frac{d}{dx}\,.$$

Exercise 6.11

We want to investigare explosion for the non-random SDE

$$dX(t) = \mu(X(t))\,dt + \sigma(X(t))\,dB(t), \quad X(0) = x_0,$$

with coefficients $\mu(x) = c\,x^r$ and $\sigma(x) = 0$, where $c, x_0 > 0$ and $r \in \mathbb{R}$ are constants. We can solve this problem by solving the SDE. Note that Feller's test for explosion Theorem 6.23 does not apply as σ is zero!

We have

$$c\,X'(t)\,X(t)^{-r} = 1 \quad \Leftrightarrow \quad \begin{cases} c\,X(t)^{1-r}/(1-r) = t+C & \text{for } r\neq 1, \\ c\,\log(X(t)) = t+C & \text{for } r=1. \end{cases}$$

Here C is an arbitrary constant. Rearranging this gives

$$X(t) = \begin{cases} X(t) = ((1-r)\,(t+C))^{1/(1-r)} & \text{for } r\neq 1, \\ X(t) = \mathrm{e}^{(t+C)/c} & \text{for } r=1. \end{cases}$$

Out of respect to the initial value $X(0) = x_0$ we conclude that

$$\begin{cases} X(t) = \big((1-r)\,t + x_0^{1-r}\big)^{1/(1-r)} & \text{for } r\neq 1, \\ X(t) = x_0\,\mathrm{e}^{t/c} & \text{for } r=1. \end{cases}$$

Now, for $r>1$ we will have an explosion at time $(1-r)\,t+x_0^{1-r} = 0$, which is to say at time $t = x_0^{1-r}/(r-1)$, On the other hand, for $r\leq 1$ there is no explosion.

Exercise 6.16

We want to find the stationary density for the SDE

$$dX(t) = \mu(X(t))\,dt + \sigma(X(t))\,dB(t)$$

with coefficients $\mu(x) = -\mathrm{sign}(x)$ and $\sigma(x) = 1$ (There is a misprint in the book, so that the sign of μ is wrong!)

By formula (6.69) we have that

$$\pi(x) = \frac{C}{\sigma(x)^2}\,\exp\bigg\{\int_{x_0}^{x} \frac{2\,\mu(y)}{\sigma(y)^2}\,dy\bigg\},$$

where the constant $C>0$ and x_0 are selected so that π has total mass 1, provided that such a selection is possible. Picking $x_0=0$ and inserting the given μ and σ, we get

$$\pi(x) = C\,\exp\bigg\{-\int_0^x 2\,\mathrm{sign}(y)\,dy\bigg\} = C\,\mathrm{e}^{-2\,|x|} = \mathrm{e}^{-2\,|x|},$$

as the function to the right integrates to 1. (Thus there is another misprint in the book: The stationary density is not $\mathrm{e}^{-|x|}$!)

Prove formula (6.69)

We want to prove that the stationary density for a diffusion must be given by formula (6.69) as

$$\pi(x) = \frac{C}{\sigma(x)^2}\,\exp\bigg\{\int_{x_0}^{x} \frac{2\,\mu(y)}{\sigma(y)^2}\,dy\bigg\},$$

where $C > 0$ and x_0 are constants, provided that π really exists.

First note that the above formula really only is the solution to the ODE

$$L_x^\star \pi(x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}\Big(\sigma(x)^2\,\pi(x)\Big) - \frac{\partial}{\partial x}\Big(\mu(x)\,\pi(x)\Big) = 0.$$

So it is enough to show that π must satisfy this ODE. However, we know that the transition density $p(t,x,y) = f_{X(t)|X(0)}(y|x)$ satisfies the Kolmogorov forward PDE

$$\frac{\partial}{\partial t}p(t,x,y) = L_y^\star p(t,x,y)$$

[see (6.32) in the book]. If π satisfies the equation for a stationary density

$$\pi(y) = \int_{-\infty}^{\infty} p(t,x,y)\,\pi(x)\,dx$$

[see (6.67) in the book], it thus follows that

$$\begin{aligned}
L_y^\star \pi(y) &= \int_{-\infty}^{\infty} L_y^\star p(t,x,y)\,\pi(x)\,dx \\
&= \int_{-\infty}^{\infty} \frac{\partial}{\partial t}p(t,x,y)\,\pi(x)\,dx \\
&= \frac{\partial}{\partial t}\int_{-\infty}^{\infty} p(t,x,y)\,\pi(x)\,dx \\
&= \frac{\partial}{\partial t}\,\pi(y) \\
&= 0!
\end{aligned}$$

Gaussian Exercise

In this exercise we want to prove that two process values $X(r)$ and $X(s)$ of a Gaussian stochastic process $\{X(t)\}_{t\in T}$ are independent if and only if they are uncorrelated.

Remember that from Lecture 4 of Part I of the course, we know that the finite dimensional distributions of a Gaussian process are determined by the mean function $m_X(t) = \mathbf{E}\{X(t)\}$ and covariance function $r_X(s,t) = \mathbf{Cov}\{X(s),X(t)\}$ of the process. This in turn is so because any vector $\big(X(t_1),\ldots,X(t_n)\big)$ of process values is multidimensional normal distributed (by the very definition of a Gaussian process), and thus determined by the mean vector

$$\big(\mathbf{E}\{X(t_1)\},\ldots,\mathbf{E}\{X(t_n)\}\big) = \big(m_X(t_1),\ldots,m_X(t_n)\big)$$

and the covariance matrix

$$\big(\mathbf{Cov}\{X(t_i),X(t_j)\}\big)_{i,j} = \big(r_X(t_i,t_j)\big)_{i,j}.$$

Now, if two process values $X(r)$ and $X(s)$ are uncorrelated, then their covariance matrix is a diagonal matrix with diagonal elements given by the variances of the process values. This in turn means that their covariance matrix is the same as if $X(r)$ and $X(s)$ where independent. But the joint distribution of $X(r)$ and $X(s)$ is determined by the mean vector and their covariance matrix, and as those quantities are the same as for $X(r)$ and $X(s)$ independent it follows that $X(r)$ and $X(s)$ are independent!