Stochastic Calculus Part II Fall 2009

Hand in 1

1. A process $\{X_t\}_{t\geq 0}$ is called *weakly stationary* if

1. $m_t := \mathrm{E}[X_t] = m \in \mathbb{R}, \forall t \geq 0$ (are all well defined),
2. Let $h \in \mathbb{R}$ be fixed. $\gamma(X_t, X_{t+h}) := \mathrm{Cov}(X_t, X_{t+h}) = \mathrm{Cov}(X_h, X_0)$ for all $t \geq 0$ (where $t + h \geq 0$ with the covariances being well defined).

A process $\{X_t\}_{t\geq 0}$ is called *stationary* if
for $n \in \mathbb{N}$, $t_1, \ldots, t_n \geq 0$, and $h \in \mathbb{R}$ where $t_1 + h, \ldots, t_n + h \geq 0$
$\big(X_{t_1}, \ldots, X_{t_n}\big) \stackrel{d}{=} \big(X_{t_1+h}, \ldots, X_{t_n+h}\big)$

Show that
i) If X_t is a stationary process, then it is also weakly stationary (assuming that all moments involved are well defined). (1 point)
ii) Assume that X_t is a Gaussian process. Show that in this case weak stationarity implies stationarity (i.e. in this case the two types of stationarity are equivalent). (1 point)

2. Let X_t be a diffusion with coefficients $\mu(x) = cx$ and $\sigma(x) = 1$.
a) Give its generator. (1 point)
b) Show that the process $X_t^2 - 2c\displaystyle\int_0^t X_s^2\, ds - t$ is a martingale. (1 point)

3. Is the *Ornstein-Uhlenbeck* (OU) process (given in example 5.6, p.127) recurrent/transient? (1 point)
Show that the limiting distribution of the OU process exists and that this distribution is given by its stationary distribution. (X_0 is treated as non-random) (Hint: What does chapter 4 say about the distribution of $\sigma e^{-\alpha t}\displaystyle\int_0^t e^{\alpha s}\, dB_s$? If you know this distribution you can take the limit of it (the distribution) to get the required limit distribution. These two distributions will belong to the same family of distributions.) (1 point)