**Stochastic Calculus Part II Fall 2009**

**Hand in 5**

Let $\{N_t\}_{t\geq 0}$ be a Poisson process with intensity $\lambda = 1$.
Let $M_t = N_t - t$ be the compensated Poisson process.

**1** Calculate $Y_t = \int_0^t M_{s-} dM_s$.
What is the quadratic covariation $[Y, M]_t$? (2 points)

**2** Let $\Lambda_t = e^{-t} 2^{N_t}$. Calculate $\mathbb{E}[\Lambda_t]$ and $Var(\Lambda_t)$.
Show that $\Lambda_t$ is a martingale.
Simulate a trajectory of $\Lambda_t$. (2 points)

**3** As you know the Compound Poisson Process (CPP) $\{X_t\}_{t\geq 0}$ can be defined as

$$X_t = X_0 + \sum_{i=1}^{\infty} \mathbb{I}(\tau_i \leq t)\xi_i = X_0 + \sum_{i=1}^{N_t} \xi_i$$

where the $\xi_i$ are iid random variables, independent of $\{N_t\}_{t\geq 0}$ (which has jump times $\tau_i$).
Let $X_0 = 0$ and $\xi_i$ be iid $N(0,1)$.

a) Show, from $E[X_t|\mathcal{F}_s] = X_s$, that $X_t$ is a martingale. (1 point)

b) Calculate $E[X_t]$ and $Var(X_t)$. (1 point)

c) Find estimates of $E[X_t]$ and $Var(X_t)$ by doing simulations. (1 point)

d) Show that the characteristic function of $X_t$ is given by

$$\phi(u) := E[e^{iuX_t}] = e^{t(\phi_\xi(u)-1)} = e^{t(e^{-u^2/2}-1)}.$$

The fact that the characteristic function of the $\xi_i$'s is given by $\phi_\xi(u) = e^{-u^2/2}$ does not have to be shown. (1 point)

e) Find the moment generating function of $X_t$. (1 point)

f) Calculate $E[X_t^4]/3t$. (1 point)

**4** The Weibull distribution is widely used for modelling life times of objects. Its density and cdf, for $\alpha, \beta > 0$ and $\nu \in \mathbb{R}$, are given by

$$f(x;\alpha,\beta,\nu) = \begin{cases} 0 & x \leq \nu \\ \dfrac{\beta}{\alpha}\Big(\dfrac{x-\nu}{\alpha}\Big)^{\beta-1} \exp\Big\{-\Big(\dfrac{x-\nu}{\alpha}\Big)^{\beta}\Big\} & x > \nu \end{cases}$$

and

$$F(x;\alpha,\beta,\nu) = \begin{cases} 0 & x \leq \nu \\ 1 - \exp\Big\{-\Big(\dfrac{x-\nu}{\alpha}\Big)^{\beta}\Big\} & x > \nu \end{cases}$$

respectively (where $\beta$ is the shape parameter, $\alpha$ is the scale parameter, and $\nu$ is the location parameter).

Just as in the CPP case let $\{X_t\}_{t\geq 0}$ be given by

$$X(t) = X(0) + \sum_{n=0}^{\infty} \mathbb{I}(\tau_i \leq t)\, \xi_i,$$

However, now let the interarrival times $\tau_{i+1} - \tau_i$ be iid Weibull$(\alpha, \beta, \nu)$ = Weibull$(\alpha, 1, 0)$-distributed, for some $\alpha > 0$, and let the jumps $\xi_i$ be iid N$(\mu, \sigma)$-distributed.

a) Find the compensator $A(t)$ of $X(t)$. (1 point)

b) Simulate one trajectory of $X(t)$ and one of $X(t) - A(t)$.
Also simulate estimates of $E[X(t)]$ and $E[M(t)] = E[X(t) - A(t)]$. (1 point)

c) Give an example of a possible situation where one could use $X(t)$ as a modelling tool. (1 point)