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# Basics of Probability and Statistics

## Probability theory

mathematical study of uncertainty and random variation

1. Probability rules
2. Discrete random variables
3. Continuous random variables
4. Joint distributions
5. Limit theorems

## Mathematical statistics

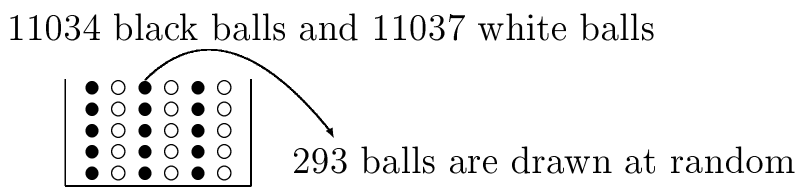
deals with variation in data using probability theory

6. Parameter estimation
7. Hypotheses testing
8. Simple linear regression

## Ex 1: aspirin treatment

Is heart attack risk reduced by taking aspirin?

11034 took placebo and 11037 took aspirin: of them
189 and 104 subsequently experienced heart attacks



# 1. Probability rules

random experiment → random event → probability

## 1.1 Random events

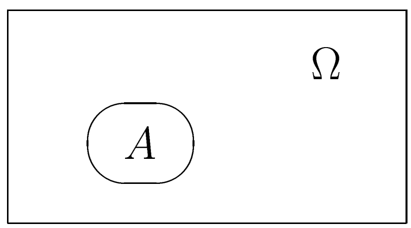
Informally: a random even $A$ either occurs or not
in a random experiment $\Omega$

**Def 1: sample space**

$\Omega$ is the set of all possible outcomes in a random experiment (finite or infinite, discrete or continuous)

**Def 2: random event**

$A$ is a subset of $\Omega$, $A \subset \Omega$




Venn diagram

**Ex 2: birthday problem**

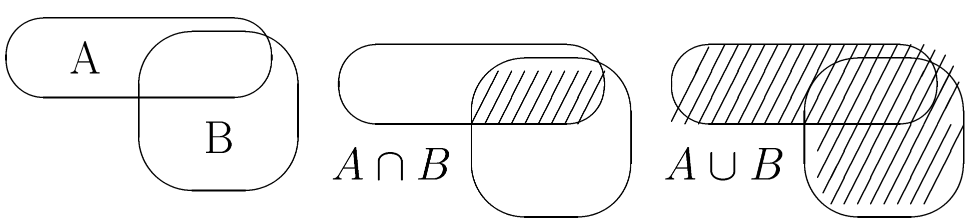
Experiment: 36 students' birthdays

$\Omega = \{(i_1, \ldots, i_{36})\}_{1 \le i_1 \le 365, \ldots, 1 \le i_{36} \le 365}$

event $A = \{\text{at least two of 36 have a common birthday}\}$

**Def 3: intersection and union of two events**

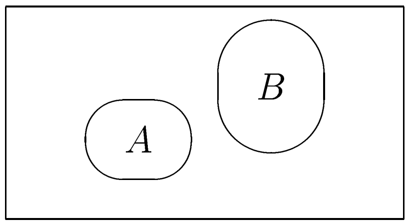
$A \cap B = \{A \text{ and } B\}$, $A \cup B = \{A \text{ or } B \text{ or both}\}$

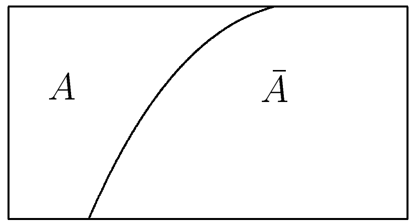


**Def 4: mutually exclusive (disjoint) events**

$A$ and $B$ are mutually exclusive, if $A \cap B = \emptyset$

complementary event to $A$: $\bar{A} = \{A \text{ does not occur}\}$




## 1.2 Probability

Informally: probability $\mathrm{P}(A)$ of the event $A$ is a number between 0 and 1 saying how likely $A$ is to occur

$\mathrm{P}(A) = 1$ means $A$ is certain

$\mathrm{P}(A) = 0$ means $A$ is impossible

$$\boxed{\text{probability} = \text{population proportion}}$$

**Def 5: probability measure**

$\mathrm{P}(\cdot)$ satisfies three axioms

1. $\mathrm{P}(\Omega) = 1$
2. if $A \in \Omega$, then $\mathrm{P}(A) \geq 0$
3. if $A$ and $B$ are disjoint, $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$

**Addition rule of probability**

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$\mathrm{P}(A) + \mathrm{P}(\bar{A}) = 1$, $\mathrm{P}(\bar{A}) = 1 - \mathrm{P}(A)$

$\mathrm{P}(A \cup B \cup C) = \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C)$

$- \mathrm{P}(A \cap B) - \mathrm{P}(A \cap C) - \mathrm{P}(B \cap C) + \mathrm{P}(A \cap B \cap C)$

## 1.3 Division rule

$$\boxed{\begin{array}{c}\text{Division rule: if all outcomes are equally likely, then}\\ \mathrm{P}(A) = \frac{\#(A)}{\#(\Omega)} = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}\end{array}}$$

**Ex 3: two dice experiment**

Two dice are rolled: $\#(\Omega) = 6 \times 6 = 36$

$\mathrm{P}(\text{the sum of points on two dice equals } 5) = \frac{4}{36} = \frac{1}{9}$

**Ex 4: sibling sampling**

Five families with two children:

three with boy and girl, two with boy and boy

Two sampling experiments

experiment 1: pick a family at random

experiment 2: pick a boy at random, consider his family

find $\mathrm{P}(A)$, $A = \{\text{the chosen family has two boys}\}$

## 1.4 Basic combinatorics

How to count the numbers of outcomes $\#(\Omega)$

in an $r$-step experiment given

$N_i = \#(\text{outcomes in the } i\text{-th step})$, tree of outcomes

$$\boxed{\text{Multiplication principle: } \#(\Omega) = N_1 \times N_2 \times \ldots \times N_r}$$

**Ex 5: sampling with replacement**

Random experiment:

draw $n = 3$ balls with replacement from a box

containing $N = 4$ balls labelled $\{1, 2, 3, 4\}$

$\#(\Omega) = 4 \times 4 \times 4 = 64$

**Def 6: permutation and combination**

permutation = the ordered set of labels in the sample
combination = unordered set of labels in the sample

$$\boxed{\text{Number of permutations of } N \text{ distinct objects taken } n \text{ at a time: } N \times (N-1) \times \ldots \times (N-n+1) = \tfrac{N!}{(N-n)!}}$$

$$\boxed{\text{The number of combinations of } N \text{ distinct objects taken } n \text{ at a time equals } \tbinom{N}{n} = \tfrac{N!}{n!(N-n)!}}$$

Numbers $\binom{n}{k}$ form Pascal's triangle and are often called binomial coefficients due to the expansion
$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \ldots + \binom{n}{n-1}ab^{n-1} + b^n$

**Ex 6: sampling without replacement**

Four objects are taken 3 at a time

number of permutations $= 4 \times 3 \times 2 = 24$
number of combinations $= \frac{24}{3\times2\times1} = 4$

$\boxed{123\ 132\ 213\ 231\ 312\ 321}$ $\boxed{124\ 142\ 214\ 241\ 412\ 421}$
$\boxed{134\ 143\ 314\ 341\ 413\ 431}$ $\boxed{234\ 243\ 324\ 342\ 423\ 432}$

**Ex 2: birthday problem**

Event $A = \{$at least two of 36 have a common birthday$\}$

$\#(\Omega) = 365^{36} = 1.748 \cdot 10^{92}$
$\#(\bar{A}) = 365 \cdot 364 \cdots 330 = 2.93 \cdot 10^{91}$
$\mathrm{P}(\bar{A}) = 0.17$, $\mathrm{P}(A) = 0.83$

**Def 7: multinomial coefficient**

Number of possible allocations in the random experiment:

allocate $n$ distinct objects into $r$ distinct boxes

box sizes $n_1, \dots, n_r$

total size of the boxes $n_1 + \dots + n_r = n$

$$\boxed{\text{Multinomial coefficient } \binom{n}{n_1, n_2, \dots, n_r} = \frac{n!}{n_1! n_2! \dots n_r!}}$$

In particular binomial coefficient $\binom{n}{k} = \binom{n}{k, n-k}$

$$\boxed{\begin{array}{c} (a_1 + \dots + a_r)^n = \Sigma \binom{n}{n_1, n_2, \dots, n_r} a_1^{n_1} \dots a_r^{n_r} \\ \text{sum over } n_1, n_2, \dots, n_r \text{ satisfying } n_1 + \dots + n_r = n \end{array}}$$

**Ex 7: birthmonths**

Experiment: birthmonths of 36 students

event $A = \{$3 in Jan, 3 in Feb, ..., 3 in Dec$\}$

$\#(\Omega) = 12^{36} = 7.088 \cdot 10^{38} = \Sigma \binom{36}{n_1, n_2, \dots, n_{12}}$

$\#(A) = \binom{36}{3,3,3,3,3,3,3,3,3,3,3,3} = 1.7 \cdot 10^{32}$

$\mathrm{P}(A) = 2.4 \cdot 10^{-7}$

**1.5 Conditional probability**

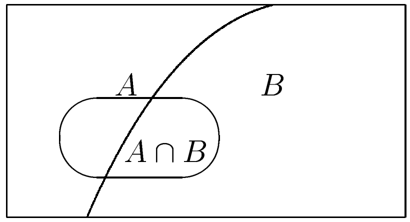
**Def 8: joint probability of two events**

$\mathrm{P}(A \cap B)$

**Def 9: conditional probability**

$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$

of a random event $A$ given that event $B$ has occurred




$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$$
$$\mathrm{P}(B|A) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)}$$

**Multiplication rule of probability**

$\mathrm{P}(A \cap B) = \mathrm{P}(A|B)\mathrm{P}(B)$

$\mathrm{P}(A \cap B \cap C) = \mathrm{P}(A|B \cap C)\mathrm{P}(B|C)P(C)$

**Ex 2: birthday problem**

Event $A = \{$at least two of 36 have a common birthday$\}$

$\bar{A} = A_1 \cap A_2 \cap \ldots \cap A_{36}$

$\mathrm{P}(\bar{A}) = \frac{365}{365} \cdot \frac{364}{365} \cdots \frac{330}{365} = 0.17$

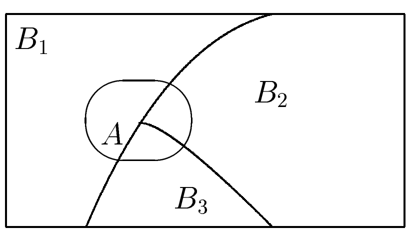
$\mathrm{P}(A) = 0.83$

**Def 10: partition** $\{B_1, B_2, B_3\}$ of $\Omega$

pairwise mutually exclusive events, $B_1 \cup B_2 \cup B_3 = \Omega$

**The Law of Total Probability (LTP)**

$\mathrm{P}(A) = \mathrm{P}(A|B)\mathrm{P}(B) + \mathrm{P}(A|\bar{B})\mathrm{P}(\bar{B})$




$\mathrm{P}(A) = \mathrm{P}(A|B_1)\mathrm{P}(B_1)$
$+\mathrm{P}(A|B_2)\mathrm{P}(B_2) + \mathrm{P}(A|B_3)\mathrm{P}(B_3)$

Given a partition $\{B_1, B_2, B_3\}$ of $\Omega$

$\mathrm{P}(A) = \mathrm{P}(A|B_1)\mathrm{P}(B_1) + \mathrm{P}(A|B_2)\mathrm{P}(B_2) + \mathrm{P}(A|B_3)\mathrm{P}(B_3)$

**Ex 8: coin-die experiment**

first step: a fair coin is tossed: $\mathrm{P}(H) = \frac{1}{2}$, $\mathrm{P}(T) = \frac{1}{2}$

second step: a die is rolled once if $H$ or twice if $T$

Tree of outcomes: $6+36 = 42$ not equally likely outcomes

$D = \{\text{total die score}\}$, random event $\{D = 5\}$

Division rule:

$\mathrm{P}(D = 5|H) = \frac{1}{6}$, $\mathrm{P}(D = 5|T) = \frac{4}{36}$

Multiplication rule: joint probabilities

$\mathrm{P}(D = 5, H) = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$

$\mathrm{P}(D = 5, T) = \frac{1}{9} \cdot \frac{1}{2} = \frac{1}{18}$

LTP

$\mathrm{P}(D = 5) = \frac{1}{12} + \frac{1}{18} = \frac{5}{36} = 0.139$

**Ex 9: three door puzzle**

Three doors with invisible numbers 1, 2, 3

a car behind door 1, nothing behind doors 2 and 3

Step 1. You point randomly at a door number $X_1$

$\mathrm{P}(X_1 = 1) = \mathrm{P}(X_1 = 2) = \mathrm{P}(X_1 = 3) = 1/3$

Step 2. A door number $X_2 \neq X_1$ is opened for you

the is no car behind it

Step 3. You are free to choose between door $X_1$

and the other unopened door $X_3$

Compair

$\mathrm{P}(X_1 = 1|X_2 = 2 \cup X_2 = 3)$ and

$\mathrm{P}(X_3 = 1|X_2 = 2 \cup X_2 = 3)$

## 1.6 Bayes' formula

$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$

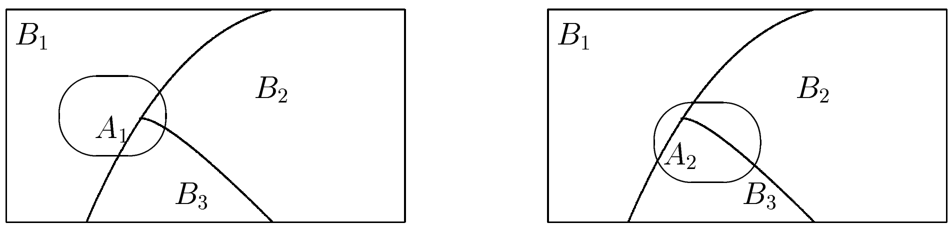
For any partition $\{B_1, \ldots, B_k\}$ of the sample space $\Omega$

$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_{j=1}^{k} P(A|B_j)P(B_j)}$

## Def 11: prior and posterior probabilities

$P(B)$ the probability of $B$ before a measurement

$P(B|A)$ the probability of $B$ after $A$ is observed




Events $A_1$ and $A_2$ allocate differently
the posterior probabilities for $\{B_1, B_2, B_3\}$

## Ex 8: coin-die experiment

Prior probabilities for the coin experiment

$P(H) = \frac{1}{2}$, $P(T) = \frac{1}{2}$

Posterior probabilities

$P(H|D = 5) = \frac{1/12}{5/36} = 3/5$

$P(T|D = 5) = \frac{1/18}{5/36} = 2/5$

## 1.7 Independence

**Def 12: independent events**

Events $A$ and $B$ are called independent if knowing that one event has occured gives no information about the other event: $\mathrm{P}(A|B) = \mathrm{P}(A)$ and $\mathrm{P}(B|A) = \mathrm{P}(B)$

$$\boxed{A \text{ and } B \text{ are independent if } \mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)}$$

**Def 13: mutually independent events**

$A$, $B$, $C$ are mutually independent if they are pairwise independent and $\mathrm{P}(A \cap B \cap C) = \mathrm{P}(A)\mathrm{P}(B)\mathrm{P}(C)$

**Ex 2: birthday problem**

Experiment: ask $n$ people for their birthday
event $B_n = \{\text{all } n \text{ birthdays different from yours}\}$

Independent events
$A_i = \{\text{person number } i \text{ has a different birthday}\}$
with equal probabilities $\mathrm{P}(A_i) = 364/365$
$\mathrm{P}(B_n) = \mathrm{P}(A_1 \cap \ldots \cap A_n) = (364/365)^n$

You must ask $n = 253$ people
to have 0.5 probability of the same birthday

**Ex 10: three coins**

Toss two fair coins, then for the third coin
choose $H$ if two heads or two tails
choose $T$ if one heads one tails

The three coin outcomes are pairwise independent
despite mutual dependence: $\mathrm{P}(T_1 \cap T_2 \cap T_3) = 0$