Modelling financial assets with stochastic processes

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1 Introduction

In financial mathematics, the modelling of financial assets using stochastic processes is a fundamental issue. Let us denote by $S(t)$ the price of for example a stock at time t (in this text, t is measured in days). Over the years several models have been proposed for $S(t)$. The most classical and widely used model is the so called Bachelier-Samuelson model, which is given by

$$S(t) = S(0)\mathrm{e}^{\mu t + \sigma W(t)} \tag{1}$$

where $S(0)$ is the price at time 0, $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}_+$, and $W(t)$ is a standard Wiener process, also known as Brownian motion. The parameter μ is called *drift* and σ^2 is called *volatility*.

Exercise 1 Let $t > s$. Show that the distribution of $\log(S(t)/S(s))$ is $N(\mu(t-s), \sigma^2(t-s))$.

Exercise 2 Calculate $\mathbf{E}\{S(t)\}$.

Exercise 3 Why is it a better idea to consider the model $S(t) = S(0)\mathrm{e}^{\mu t + \sigma W(t)}$ than the simpler model $S(t) = \mu t + \sigma W(t)$?

The reason the Bachelier-Samuelson model is so popular is that the theory for the Wiener process, which is the most important of all Lévy processes, is very well developed. From the Bachelier-Samuelson model arises many nice formulae in the theory of *option pricing*, see for example [4]. However, it is known through many empirical studies that the this model is not a realistic model. In fact, it has many unwanted properties some of which are listed below.

1. The distribution of the increments of $\log(S(t))$ has very light tails.
2. The increments of $\log(S(t))$ over disjoint time intervals are independent.
3. The increments of $\log(S(t))$ have a symmetric distribution.

The first of these means that the probability of a big change in the price of the stock overnight is very small. In reality, big changes do happen not very seldom. The second disadvantage means that the change in the price for a day does not depend on the change in the price the day before. In reality however, it is known that a big change is often followed by another big change. The third disadvantage will be visualized later.

To get rid of some (but not all) of these disadvantages, it is possible to consider a more general model, namely

$$S(t) = S(0)\mathrm{e}^{X(t)} \tag{2}$$

where $X(t)$ is a Lévy process (note that the Bachelier-Samuelson model is included in this class of models).

Exercise 4 Which of the unwanted properties for a model of the price of a stock cannot be avoided by this model? Motivate your answer!

In the last few years, the so called Normal Inverse Gaussian (NIG) process has been the subject of many studies in empirical finance. It has turned out that this process is a good choice for the process $X(t)$ above. Before defining the NIG process, we define the Normal Inverse Gaussian distribution.

Definition 1 (One-dimensional NIG distribution) *A one-dimensional NIG distribution has the following density function*

$$f_{NIG}(x;\alpha,\beta,\delta,\nu) = \frac{\alpha\delta}{\pi}\exp\left(\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\nu)\right)\frac{K_1(\alpha\sqrt{\delta^2+(x-\nu)^2})}{\sqrt{\delta^2+(x-\nu)^2}}$$

for $x \in \mathbb{R}$, where $\nu \in \mathbb{R}$, $\delta \geq 0$, $0 \leq |\beta| \leq \alpha$, and K is the modified Bessel function of the third kind.

Here, ν is a location parameter, β a skewness parameter (affects the shape of the density), and α and δ are scale parameters. A NIG process is a Lévy process $\{X(t)\}_{t\geq 0}$ such that the distribution of $X(1)$ is given by the above defined NIG-distribution. It is not difficult to show that if $\{X(t)\}_{t\geq 0}$ is a NIG process, then the density of $X(t)$ is given by

$$f_{X(t)}(x) = f_{NIG}(x;\alpha,\beta,\delta t,\nu t). \tag{3}$$

From now on we say that $X \sim NIG(\alpha,\beta,\delta,\nu)$ if X has a NIG distribution with parameters α, β, δ and ν.

Exercise 5 If $\{X(t)\}_{t\geq 0}$ is a NIG process, show that if $t > s$ then $X(t) - X(s) \sim NIG(\alpha,\beta,\delta(t-s),\mu(t-s))$

2 Estimation of model parameters

Suppose we are given a time series of for example stock prices for n days which we denote by $\{x_i\}_{i=1}^n$. We define the series of *logreturns* by $\{r_i\}_{i=1}^{n-1} = \{\log(x_{i+1}/x_i)\}_{i=1}^{n-1}$. Modelling the stockprice according to (1) means that we consider $\{r_i\}_{i=1}^{n-1}$ as $n-1$ independent observations from a $N(\mu, \sigma^2)$-distribution according to the exercise. If we instead use the model (2) then we consider $\{r_i\}_{i=1}^{n-1}$ as n independent observations from a $NIG(\alpha, \beta, \delta, \mu)$-distribution. We now want to estimate the parameters in the models, and to do this we will use the so called *maximum likelihood method*, which you might remember from you first statistics course.

2.1 Maximum likelihood method

If we are given n independent observations $\{x_i\}_{i=1}^n$ of a random variable X with density function $f_X(x; \theta_1, ..., \theta_n)$ where $\theta_1, ..., \theta_k$ are parameters, the *(observed) maximum likelihood estimate* of $\theta_1, ..., \theta_k$ is given by

$$(\hat{\theta}_1, ..., \hat{\theta}_k) = \text{argmax}_{\theta_1,...,\theta_k} \prod_{i=1}^n f_X(x_i; \theta_1, ..., \theta_k) \tag{4}$$

The product in (4) is called the *likelihood function*, and the maximum likelihood estimate is simply those parameter values maximizing the likelihood function. Quite often it is more convenient to maximize the natural logarithm of the likelihood function, which gives the same estimate since the natural logarithm is an increasing monotone function. When maximizing the likelihood function, we (in some sense) choose the parameters making the observations most likely (this description is only completely correct in the discrete case, (why?)). In the case of estimating the parameters μ and σ in the normal distribution, the maximum likelihood estimates are explicitly known:

$$\hat{\mu} = \frac{\sum_{i=1}^n x_i}{n} \tag{5}$$

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^n (x_i - \hat{\mu})^2}{n}. \tag{6}$$

In general, one does not have such explicit formulas, for example in the NIG case we do not. In this case, we numerically have to optimize the likelihood function (4) with respect to α, β, δ, ν numerically using for example the softwares `MatLab` or `Mathematica`.

2.2 Computer task

In the internet directory `http://www.math.chalmers.se/~johant/nigfiles/` you can find three files for this task. The file `ericsson.txt` contains returns for the Ericsson stock between September 2000 and September 2002 (you have to take the log of the data to get the logreturns). Now fit the model parameters μ and σ^2 in the Bachelier-Samuelson model to the data. Also fit the logreturns to a $NIG(\alpha, \beta, \delta, \nu)$-distribution. To do this, one can use the `Matlab` routine `fminsearch` or the `Mathematica` routine `FindMinimum`. These routines are so called optimization routines, using different algorithms trying to find where a function has its smallest value. Such algorithms needs to have starting values specified, that is, they need to know where to start looking for the smallest value. In this case, one has to give some more or less good guess about what the NIG-parameters might be. This can be done for example by looking in some article where NIG-parameters have been estimated for some other stock price. If unfamiliar with optimization in general, the optimization routines might not be so easy to handle. Since this project is aimed at understanding the modelling of asset prices by stochastic processes, rather than `Matlab` or `Mathematica` programming, you can use the `Matlab` files `liknig.m` and `skattning.m` in the directory above to do the optimization. To understand how they work, you need to read in the `Matlab` manual about `fminsearch`.
Using the estimated parameters, now plot and compare the normal density and the NIG-density.

3 Testing the fit of the models

So now we have estimated model parameters in two different models for the Ericsson stock. However, we do not know how good our models fit to the data. The so called "goodness of fit" can be examined in different ways. One can calculate different test statistics or look at graphs. Here we choose to do the latter, since it can be used without introducing too much new theory. What we will do is to compare the so called empirical distribution function with the estimated distribution functions. The estimated distribution functions are

$$F_{NIG}(x; \hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\nu}) = \int_{-\infty}^{x} f_{NIG}(y; \hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\nu})\, dy \tag{7}$$

and

$$F_{Normal}(x;\hat{\mu},\hat{\sigma}^2) = \int_{-\infty}^{x} f_{Normal}(y;\hat{\mu},\hat{\sigma}^2)\,dy \tag{8}$$

where $\hat{\alpha}$, $\hat{\beta}$, $\hat{\delta}$, $\hat{\nu}$, $\hat{\sigma}^2$ and $\hat{\mu}$ are our estimated parameters. Given observations $\{x_i\}_{i=1}^n$ from some random variable X the *(observed) empirical distribution function* is defined in the following way:

$$\hat{F}_{emp}(x) = |\{x_i : \, x_i \leq x, \, 1 \leq i \leq n\}|/n \tag{9}$$

where $|\{...\}|$ denotes the number of elements of the set $\{...\}$.

In other words, the empirical distribution function at x is the percentage of observations smaller than or equal to x. Plot the empirical distribution function for the logreturns of the Ericsson data set and compare with the estimated distribution functions in the normal and the NIG-case. Which one of the estimated distribution functions looks most like the empirical one?

4 Simulation

Unfortunately, simulation of a NIG-process is beyond the scope of this course. But we can easily simulate $S(t)$ in the Bachelier-Samuelson case, since a Wiener process is easily simulated. For hints how to do this, see chapter 5 in [3]. Simulate some trajectories of $S(t)$ with the estimated μ and σ^2. What happens if one varies μ and σ? Good luck!

References

[1] Johan Tykesson, *Some Aspects of Lévy processes in finance*, Master's thesis, Chalmers University of Technology, 2003.

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[3] Patrik Albin, a.k.a the King of Stochastic Processes, *Stokastiska processer*, Studentlitteratur, 2003.

[4] Christer Borell, *Financial Derivatives and Stochastic Analysis*, Chalmers University of Technology, 2001.