Home exam in **Randomized algorithms**

The exam consists of five problems, each being worth four points, so that the total number of points is 20. 10 points are required to pass the exam.

Deadline, for returning the solutions, is Thursday, May 30. It can be handed in to Lotta Fernström (room 1403 at Matematiskt Centrum, her office hours on May 30 are 9.00-12.30 and 13.00-15.00), or you can send it to me directly at address *Olle Häggström, Institut Mittag-Leffler, Auravägen 17, 182 60 Djursholm* (poststämpel avgör!). For other rules concerning this exam, see
`http://www.math.chalmers.se/~olleh/randomiserade_algoritmer.html`

1. This problem is about understanding and applying the definition of a Markov chain. Let $\ldots, X_{-1}, X_0, X_1, X_2, \ldots$ be an i.i.d. sequence of random variables taking values 0 or 1 with probability $\frac{1}{2}$ each.

   (a) Is $(X_0, X_1, \ldots)$ a Markov chain?

   (b) For each i, let $Y_i = X_{i-1} + X_i$. Is $(Y_0, Y_1, \ldots)$ a Markov chain?

   (c) For each i, let $Z_i = (X_{i-1} + X_i) \bmod 2$. (In other words, $Z_i = 0$ if $X_{i-1} = X_i$, and $Z_i = 1$ otherwise.) Is $(Z_0, Z_1, \ldots)$ a Markov chain?

2. **Ehrenfest's urn model.** Fix an integer k, and imagine two urns, each containing a number of balls, in such a way that the total number of balls in the two urns is k. At each integer time, we pick one ball at random (each with probability $\frac{1}{k}$) and move it to the other urn[1]. If X_n denotes the number of balls in the first urn, then $(X_0, X_1, \ldots)$ forms a Markov chain with state space $\{0, \ldots, k\}$.

   (a) Write down the transition matrix of this Markov chain.

   (b) Show that the Markov chain is reversible with stationary distribution $\pi = (\pi_0, \ldots, \pi_k)$ given by

   $$\pi_i = \frac{k!}{i!(k-i)!} 2^{-k} \quad \text{for } i = 0, \ldots, k\,.$$

   (c) Show that the same distribution (known as the **binomial distribution**) also arises as the distribution of a binomial $(k, \frac{1}{2})$ random variable, as defined in Example 1.3 of *"Finite Markov chains and algorithmic applications"*.

   (d) Can you give an intuitive explanation for why Ehrenfests's urn model and Example 1.3 give rise to the same distribution?

[1] There are various interpretations of this model. Ehrenfest's original intention was to model diffusion of molecules between the two halves of a gas container.

3. Suppose that we have a finite set S with n elements (n large), and an **integer-valued** function $f : S \to \mathbf{N}$ whose minimum we want to find.

   (a) Find a temperature $T_c > 0$ small enough so that for all $T < T_c$, the Boltzmann distribution $\pi_{f,T}$ is guaranteed to place probability at least 0.99 (say) on the set of f-minimizers in S. (The answer depends on n.)

   (b) Discuss whether the answer to (a) should have any particular consequences for the choice of cooling schedule when we use simulated annealing to find the minimum of f.

4. Many tasks in computer science take the shape of a question that just requires a simple YES/NO answer. Suppose that we have such a task (whose answer depends on some input), and a randomized algorithm A with the properties that (i) the running time grows at most polynomially in the size of the input, and (ii) for any input, the probability of producing the correct answer is at least 0.501.

   (a) Show how to, given the algorithm A, construct another algorithm A^* which also runs in polynomial time, and which produces a correct answer with probability at least 0.999.

   (b) Can a similar result be obtained if the probability in (ii) is 0.5 rather than 0.501?

5. Consider the Gibbs sampler for the hard-core model, as described in Example 7.2 of *"Finite Markov chains and algorithmic applications"*. A couple of years ago, it was proved that if we restrict to graphs of degree at most 4 (i.e., every vertex has at most 4 neighbors), then the time taken for the distribution $\mu^{(n)}$ at time n to come within total variation distance ε from the target distribution μ_G, is at most $Ck^3 \log(\varepsilon^{-1})$. Here k is the number of vertices in the graph, and C is a constant (not depending on k or on ε).

   Use this result to show that there exists a randomized polynomial time approximation scheme for counting the number of feasible hard-core configurations of such graphs.