HOME EXAM IN RANDOMIZED ALGORITHMS

2003.05.22-2003.05.30

The exam consists of 5 problems, each being worth 4 points. To pass the exam 10 points are required. Deadline for returning solutions is Friday, May 30, 16.00, at the Department of Mathematical Statistics, Eklandagata 86. For other rules regarding the exam see the course home page

`www.math.chalmers.se/Stat/Grundutb/Chalmers/TMS081/`

Solutions should be easy to read and follow, motivate used theorems, and **do not** forget to answer the questions asked. Finally . . .

GOOD LUCK !!!

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1. A Markov chain application Let $X_0, X_1, X_2, ...$ be a sequence of i.i.d. random variables taking values in the set $\{1, 2, 3\}$ with probability $1/3$ each, and let $S_k = X_0 + X_1 + ... + X_k$ for $k = 0, 1, 2, ...$ In the long run, how many times will the sum S_k be divisible by 5? Use a Markov chain to solve this problem.

2. Functions of Markov chains Let $X_0, X_1, ...$ be a Markov chain on the state space, $S = (s_1, s_2, s_3)$ with transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and initial distribution $\mu^{(0)} = (1/3, 1/3, 1/3)$. Let for each k

$$Y_k = \begin{cases} 1, & \text{if } X_k = s_1 \\ 0, & \text{otherwise} \end{cases}$$

Is $Y_0, Y_1, ...$ a Markov chain? Explain and show!

3. An MCMC algorithm for the Potts model An extension of the Ising model is the Potts model, instead of having just two states for each edge we now have q states. Let the graph $G = (V, E)$ be a box in $\mathbb{Z}^2$,

$$V = \mathbb{Z}^2 \cap [-n, n] \ , \quad E = \{\langle x, y \rangle : x, y \in V, \ |x - y| = 1\}$$

where $|\cdot|$ is the Euclidean distance for the plane, and n is some finite integer. In the Potts model each vertice is assigned one state in the set $\{1, 2, ..., q\}$, depedning on the state of it's neighbours. Such an assignment, $\omega \in \Omega = \{1, 2, ..., q\}^V$, is given probability

$$\mu_{G,q}(\omega) = \frac{1}{Z}\, e^{-\beta H(\omega)} \ , \quad H(\omega) = 2 \sum_{\langle x,y \rangle \in E} I_{\{\omega(x) \neq \omega(y)\}}$$

where Z is just a normalisation constant.

Propose an algorithm for simulating $\mu_{G,q}$, and verify that it has the desired convergence property.

4. **A decision problem** A polynomial time algorithm for a decision problem in computer science return yes/no for each instance, and has the running time bounded by a polynomial in the problem size, say n. Assume we need a randomized algorithm for such a problem with polymial running time, and we need it to give a correct answer in 999 of 1000 cases.

   We have at our disposal a polynomial time randomized algorithm A for this problem, but there is a problem, it gives a correct answer with probability p for some given $p \in [0.5, 0.9]$.

   Propose a polynomial time randomized desicion algorithm A^* for the original problem, using the given algorithm A. Can there be any problems using the given algorithm?

5. **Approximate counting** Consider the Gibbs sampler for the hard-core mode, as described in example 7.2 in *"Finite Markov chains and applications"*. A couple of years ago, it was proved that if we restrict to graphs of degree at most 4 (i.e. every vertex has at most 4 neighbours), then the time taken for the distribution $\mu^{(n)}$ at time n to come within total variation distance ε from the target distribution μ_G, is at most $Ck^3 \log(1/\varepsilon)$. Here k is the number of vertices in the graph, and C is a constant independent of k and ε.

   Use this result to show that there exists a randomized polynomial time approximation scheme for counting the number of feasible hard-core configurations of such graphs.