HOME EXAM IN RANDOMIZED ALGORITHMS

2004.05.24-2004.06.03

The exam consists of 5 problems, each being worth 4 points. To pass the exam 10 points are required. Deadline for returning solutions is Thusday, June 3, 12.00, at the Department of Mathematical Statistics, Eklandagatan 86. For other rules regarding the exam see the course home page

`www.math.chalmers.se/Stat/Grundutb/Chalmers/TMS081/`

Solutions should be easy to read and follow, motivate used theorems, and **do not** forget to answer the questions asked. Finally ...

GOOD LUCK !!!

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1. Markov chain theory

Let $(X_0, X_1, ...)$ be a Markov chain with state space $S = \{s_1, ..., s_k\}$ and transition matrix P. Suppose P is doubly stochastic. What can we say about the asymptotic distribution of the chain? Determine if the chain ...

(a) ... has a stationary distribution, and if it does, specify it.

(b) ... is irreducible.

(c) ... is aperiodic.

(d) ... has convergence towards a stationary distribution.

For each property, a proof or counterexample is required.

2. General Markov Chain Monte Carlo

Let V and S be two finite sets and let π be a probability distribution on S^V. We call an element $\xi \in S^V$ a configuration. For any $w \in V$ we let $\xi(w)$ denote the configuration at w and we let $\xi^w \in S^{V\setminus\{w\}}$ denote the configuration on every element in V except w.

When constructing a Gibbs sampler to generate samples from π we update a configuration ξ to ξ' according to the following procedure.

1. Pick an element $v \in V$ uniformly at random.
2. Let $\xi'(v) = a$ with probability $\pi(\xi(v) = a|\xi^v)$ for any $a \in S$.
3. For $w \neq v$ let $\xi'(w) = \xi(w)$

Show that this in general makes π a reversible distribution.

3. An MCMC Application

A simple version of the so called Widom-Rowlinson model is defined as follows. Let G be a finite graph (for instance, a finite portion of the square lattice), with vertex set V. An assignment $\xi \in \{-1, 0, +1\}^V$ of -1's, 0's, and +1's to the vertices of G, is said to be WR-feasible if $\xi(u)\xi(v) \geq 0$ for all pairs of vertices that share an edge. In other words, ξ is WR-feasible if no edge has a -1 at one end point and a +1 at the other. The Widom-Rowlinson model now arises by picking a WR-feasible assignment at random, with equal probability for each such assignment. Write μ_G for the corresponding probability distribution on $\{-1, 0, +1\}^V$.

Propose a sensible MCMC algorithm for simulating μ_G, and verify that it has the desired convergence property.

4. Sampling of Markov chains

Let $(X_0, X_1, ...)$ be a Markov chain with transition matrix P.

(a) Define $(Y_0, Y_1, ...)$ by setting $Y_n = X_{2n}$ for each n. Show that $(Y_0, Y_1, ...)$ is a Markov chain with transition matrix P^2.

(b) Find an appropriate generalization of the result in (a) to the situation where we sample every k^{th} (rather than every second) value of $(X_0, X_1, ...)$.

5. Convergence of Markov chains

Let $(X_0, X_1, ...)$ be an irreducible and aperiodic Markov chain with state space $S = \{s_1, ..., s_k\}$ and transition matrix P. In general P has k eigenvalues and k left eigenvectors. Let $\lambda_1, \lambda_2, ..., \lambda_k$ denote the eigenvalues in increasing order. If P is stochastic the largest eigenvalue $\lambda_k = 1$.

(a) What can you say about the rows of P^n as $n \to \infty$?

(b) Which is the corresponding left eigenvector to λ_k?

(c) What can you say about the convergence of $\mu^{(n)}$ towards the stationary distribution? Do the eigenvalues have any implication for the convergence rate?

Some results and definitions from linear algebra

Stochastic and doubly stochastic matrices :

A $n \times n$ real matrix M is called stochastic if

$$\forall i \in \{1, ..., n\} : \sum_{j=1}^{n} M_{i,j} = 1.$$

If in addition

$$\forall j \in \{1, ..., n\} : \sum_{i=1}^{n} M_{i,j} = 1.$$

holds we say that M is doubly stochastic. A doubly stochastic matrix is a matrix where all rows and all columns sum up to 1.

Eigenvalues and eigenvectors of stochastic matrices :

A $n \times n$ real matrix M has in general n eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ and to each eigenvalue there is a corresponding right eigenvector v_k^r and left eigenvector v_k^l and in general $v_k^r \neq v_k^l,\ k = 1, ..., n$.

We can factorise M as follows.

$$M = V\ D\ V^{-1}$$

Where D is a diagonal matrix

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \lambda_2 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \lambda_n \end{pmatrix}$$

and V is a matrix having the eigenvectors as columns

$$V = \begin{pmatrix} v_1^r & v_2^r & v_3^r & \cdot & \cdot & \cdot & v_n^r \end{pmatrix}$$