TMS081, Randomized Algorithms , VT 2006
Home Exam

IMPORTANT INSTRUCTIONS: Deadline for returning the solutions is **Monday, May 29, 2006**. You can return solutions by e-mail to **johant@math.chalmers.se**, by ordinary mail to **Johan Tykesson, Matematisk statistik, Chalmers, 41296 Göteborg**, or by coming to my office and leaving it directly to me (if Im not in, ask someone to put it on my desk). Solutions should be clear and easy to read. **PLEASE SAVE A COPY FOR YOURSELF IF YOU DO HANDWRITTEN SOLUTIONS!** Note that there are text on two pages.

I cannot, of course, strictly forbid you to speak to each other during the home exam. But I do require that each student thinks through and solves each problem on his/her own. It is OK if one or two little hints for solutions pass from one student to the other, but it is NOT acceptable to copy directly from another student's solutions. Everyone should be prepared to defend his or her solutions, and I reserve the right to carry out an oral examination in cases of doubt (hopefully this will not happen!). Good luck!

For details about grading, see the homepage.

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1. **(Markov chains)**
   (a) Show that any Markov chain with just two states is reversible. (2p.)
   (b) Show that there are Markov chains with three states that are not reversible, by giving an example (of course you must show that your example is not reversible.) (2p.)

2. **(Markov Chain Monte Carlo)** Suppose we want to draw m samples from approximately the distribution π on the large finite set S. Also suppose we want the samples to be independent. To do this, we construct a Markov chain $(X_0, X_1, X_2, ...)$ for MCMC simulation, and run the chain for a long time n so that the distribution of X_n is sufficiently close to π. Then we take as our m samples X_n, X_{n+1}, X_{n+2},...,X_{n+m-1}. However, this does not in general give us what we want. Explain what mistake we do, and how to go around it. (3p.)

3. **(Propp-Wilson)** In the Propp-Wilson algorithm, one needs to choose a sequence of integers $(N_1, N_2, N_3, ...)$ and then run the algorithm starting from time $-N_1$, from time $-N_2$ and so on until one finds a starting time for which the algorithm terminates (a starting time for which the parallell markov chains in the algorithm have coalesced). Let $N^* = min\{n :$ the chains starting at time $-n$ coalesce by time $0\}$. Show that if we choose $(N_1, N_2, N_3, ...)$ to be $(k^0, k^1, k^2, k^3, ...)$ for some integer $k \geq 2$, then the total number of time units we need to run the markov chains is bounded by $\left(\frac{k^2}{k-1}\right) N^*$. (As noted in problem 10.1, if $(N_1, N_2, ...) = (1, 2, 3, ...)$ then the total number of time units needed equals a quadratic function of N^*.) (4p.)

4. **(Markov Chains and Simulated Annealing)** For a positive integer m, let S be the set of permutations of $(1, 2, ..., m)$. Consider the Markov chain $(X_0, X_1, X_2, ...)$ with state space S and the following transition mechanism. At each integer time n, we pick $i, j \in \{1, 2, ..., m\}$ with $i < j$ uniformly (i.e., each pair has probability $\frac{2}{m(m-1)}$ of being picked). Then the subsequence of the permutation X_n starting in the i:th element and ending in the j:th element, is reversed. More precisely, if

$$X_n = (a_1, a_2, ..., a_{i-1}, a_i, a_{i+1}, ..., a_{j-1}, a_j, a_{j+1}, ..., a_{m-1}, a_m)$$

then

$$X_{n+1} = (a_1, a_2, ..., a_{i-1}, a_j, a_{j-1}, ..., a_{i+1}, a_i, a_{j+1}, ..., a_{m-1}, a_m).$$

   (a) Show that this Markov chain is irreducible. (2p.)

   (b) Discuss the relevance of the result in (a) to the simulated annealing algorithm in Example 13.3 of *"Finite Markov chains and algorithmic applications"*. (2p.)

5. **(Markov Chain Monte Carlo)** Let G be a finite connected graph with vertex set V. We say that a configuration $\xi \in \{0, -1, +1\}^V$ of $0's$, $1's$ and $-1's$ to the vertices is feasible if $\xi(u)\xi(v) \geq 0$ for all pairs of vertices that share an edge. In other words, a configuration is feasible if no edge has a $+1$ at one of its endvertices and a -1 at the other. Let T be the set feasible configurations. Let μ_G be the probability measure on $\{0, -1, +1\}^V$ which assigns probability $1/|T|$ to each feasible configuration and probability 0 to all other configurations. Here $|T|$ denotes the number of configurations in T.

   Propose a sensible MCMC algorithm for simulating a random configuration with distribution μ_G and verify that it has the desired convergence property. (5p.)