Some problems from chapter 2

2.2 Consider the following modification of the random walker example. At each time, he or she tosses two coins. If the first comes up heads, he stays where he is, whereas if it comes up tails, he lets the second coin decide whether he should move one step clockwise or one step counterclockwise. Write down the transition matrix and draw the transition graph for this new Markov chain.

2.3 Suppose $\{X_n\}_{n=0}^{\infty}$ is a Markov chain with state space $\{s_1, s_2\}$, transition matrix $P = \begin{pmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{pmatrix}$, and initial distribution $\mu^{(0)} = (1, 0)$. Prove by induction that

$$\mu^{(n)} = \left(\frac{1}{2}(1 + 2^{-n}),\ \frac{1}{2}(1 - 2^{-n})\right)$$

for all n and investigate what happens as $n \to \infty$.

2.4 Suppose $\{X_n\}_{n=0}^{\infty}$ is a Markov chain with state space $\{s_1, s_2\}$, transition matrix $P = \begin{pmatrix} 0.5 & 0.5 \\ 0.1 & 0.9 \end{pmatrix}$ and initial distribution (1/6, 5/6). Show that $\mu^{(n)} = \mu^{(0)}$ for all n. Can you find an initial distribution for the random walker example such that we get a similar behaviour?

2.6 Suppose $\{X_n\}_{n=0}^{\infty}$ is a Markov chain with state space $\{s_1, s_2, s_3\}$, transition matrix $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ and initial distribution $\mu^{(0)} = (1/3, 1/3, 1/3)$. For each n, define $Y_n = 0$ if $X_n = s_1$ and $Y_n = 1$ otherwise. Prove that $\{Y_n\}_{n=0}^{\infty}$ is not a Markov chain.

2.7 Suppose $\{X_n\}_{n=0}^{\infty}$ is a Markov chain with transition matrix P. Define the process $\{Y_n\}_{n=0}^{\infty}$ by letting $Y_n = X_{2n}$ for each n. Show that this is a Markov chain with transition matrix P^2. Find an appropriate generalization of this result to the situation when we instead let $Y_n = X_{kn}$ for each n for an arbitrary integer $k \geq 2$.