Serik Sagitov, Chalmers Tekniska Högskola, April 18, 2002

Basics of Mathematical Statistics

1. Parameter estimation

Random sample $(X_1, \ldots, X_n)$ and a histogram

heights between 160, 165, 170, 175, 180, 185, 190

Sample mean $\bar{X} = \frac{X_1+\ldots+X_n}{n}$

estimates unknown population mean μ

no systematic error $\mu_{\bar{X}} = \mu$

Random error in $\bar{X}$ is measured by

standard error $\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$ where unknown

population standard deviation σ is estimated with

Sample standard deviation $s = \sqrt{s^2}$

sample variance $s^2 = \frac{(X_1-\bar{X})^2+\ldots+(X_n-\bar{X})^2}{n-1}$

Estimated standard error of $\bar{X}$: $s_{\bar{X}} = \frac{s}{\sqrt{n}}$

Dichotomous data

Population proportion p of females

$X = 1$ if a female, and $X = 0$ if a male

Sample count of females $Y = X_1 + \ldots + X_n$

has the binomial distribution $\mathrm{Bin}(n, p)$

with $\mu_y = np$, $\sigma_y = \sqrt{np(1-p)}$

Sample proportion $\hat{p} = Y/n = \bar{X}$

with $\mu_{\hat{p}} = p$, $\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}$

Estimated standard error of $\hat{p}$: $s_{\hat{p}} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n-1}}$

2. Normal distribution

When sample size n is large

the Z-scores: $Z = \frac{\bar{X}-\mu}{\sigma_{\bar{X}}}$ and $Z = \frac{\hat{p}-p}{\sigma_{\hat{p}}}$

and the T-scores: $T = \frac{\bar{X}-\mu}{s_{\bar{X}}}$ and $T = \frac{\hat{p}-p}{s_{\hat{p}}}$

have standard normal distribution N(0,1)

Bell-shaped curve with area α to the right of z_α

α	0.50	0.16	0.10	0.05	0.025	0.01	0.005	0.001
z_α	0.00	1.00	1.28	1.64	1.96	2.34	2.58	3.10

Diversification experiment

What would you prefer:

a) take 4500 SEK or

b) toss a coin and get 10000 SEK in case of heads

c) toss 10000 one-SEKs and collect all heads-up coins

X={the amount of money collected in the last case}

$\mu_x = 5000$, $\sigma_x = \sqrt{10000 \cdot 0.5 \cdot 0.5} = 50$

three-sigma rule: X belongs to 5000 ± 150 SEK

3. Hypotheses testing

Extrasensory perception (ESP)

Population parameter of interest

p = probability of correctly guessing the suit of a card

Two competing hypotheses on the value of p

null hypothesis $H_0 : p = 0.25$ (pure guessing)

one-sided alternative hypothesis $H_1 : p > 0.25$

Data: a subject tries to guess the suits of $n = 100$ cards
$Y =$ the number of correct guesses
A decision rule: for some *critical value* y
if $Y \geq y$, reject H_0 in favor of H_1
if $Y < y$, do not reject H_0

	Decision: accept H_0	Decision: reject H_0
H_0 is true	Correct decision	Type I error error size α
H_1 is true	Type II error error size β	Correct decision

Conflict between α and β for fixed sample size
if a blanket is too narrow for two
get a wider blanket - increase the sample size

Assymetry between H_0 and H_1
H_0 gives a simple explanation that must be discredited
in order to demonstrate some effect H_1
Type I error has graver consequences
H_0: an accused is innocent
H_0: a new drug is not as good as the old one

4. Large-sample test for proportion

Sample count $Y \in \text{Bin}(n, p)$, H_0: $p = p_0$
test statistic $Z = \frac{Y - np_0}{\sqrt{np_0(1-p_0)}}$
For a given significance level α

one-sided H_1: $p>p_0$, Rejection Region is $\{Z > z_\alpha\}$
one-sided H_1: $p<p_0$, RR is $\{Z < -z_\alpha\}$
two-sided H_1: $p\neq p_0$, RR is $\{ Z < -z_{\alpha/2}$ or $Z > z_{\alpha/2}\}$

ESP test: $n = 100$, $p_0 = 0.25$, H_1: $p>0.25$
RR is $\{Y > y\}$ where $y = 25 + z_\alpha \cdot 4.33$

α	0.10	0.05	0.01	0.001
y	30.5	32.1	35.1	38.4

For the two-sided alternative H_1: $p\neq 0.25$
RR is $\{Y < y_1$ or $Y > y_2\}$, where $y = 25 \pm z_{\alpha/2} \cdot 4.33$

α	0.10	0.05	0.01
y_1	17.9	16.5	13.8
y_2	32.1	33.5	36.2

5. P-value

How significant is the ESP experiment result $Y = 33$
is found from the observed $Z = \frac{Y-np_0}{\sqrt{np_0q_0}} = 1.85$
using the normal distribution table
One-sided P-value of the test $P = 1 - 0.9678 = 0.032$
two-sided P-value $P = 2(1 - 0.9678) = 0.064$
The smaller is P the more significant is the observed data
reject H_0 at 5% significance level in favor of H_1: $p>0.25$
do not reject H_0 at 5% level in favor of H_1: $p\neq 0.25$

P-value of the test: the smallest level at which H_0 is rejected with a given data set

6. Large-sample test for mean

Test H_0: μ=μ_0 using test statistic T=$\frac{\bar{X}-\mu_0}{s_{\bar{X}}}$

one-sided H_1: μ>μ_0, RR is $\{T > z_\alpha\}$

one-sided H_1: μ<μ_0, RR is $\{T < -z_\alpha\}$

two-sided H_1: $\mu \neq \mu_0$, RR is $\{\ T < -z_{\alpha/2}$ or $T > z_{\alpha/2}\}$

Dimensions of cuckoos' eggs

n=243 eggs. Length and breadth in mm with frequencies:

19	19.5	20	20.5	21	21.5	22	22.5	23	23.5	24	24.5	25
1	1	7	3	29	13	54	38	47	22	21	5	2

14	14.5	15	15.5	16	16.5	17	17.5	18	18.5	19
1	1	5	9	73	51	80	15	7	0	1

Length: $\bar{X}$=22.41, s=1.08, $s_{\bar{X}}$=0.069

breadth: $\bar{X}$=16.54, s=0.66, $s_{\bar{X}}$=0.042

Test H_0: μ=22.60 for the egg length

observed $T = \frac{22.41-22.60}{0.069} = -2.75$

one-sided P-value $P = 1 - 0.9978 = 0.003$

two-sided P-value $P = 2 \cdot 0.003 = 0.006$

Reject H_0: unchanged μ compared with the previous year