Written test for examination in TMS115
TMS115 Probability and Stochastic Processes, 2005-10-21, Friday, 14:00 - 18:00, M.
Lecturer and on duty: Rossitza Dodunekova, tel. 772 3534.
Time of visit 15:00 and 17:00.
Allowed material: Calculators approved by Chalmers, the handbook *Beta*.

There are 30 total points in the examination. One needs 14 points for grade 3 (to pass), 18 points for grade 4, and 24 points for grade 5.

Problem 1. A system consists of one controller and three peripheral units. The system is said to be "up" if the controller and at least two of the peripherals are functioning. The controller fails with probability p, and each peripheral fails with probability r. Find the probability that the system is "up", assuming that all components fail independently. (2)

Problem 2. X and Y are random variables such that $X \sim U(0, 1)$ and $Y = \dfrac{-\ln X}{\lambda}$, where $\lambda > 0$ is a constant.

(a) Find the pdf of Y. (2)

(b) Compute $E[XY]$. (2)

Problem 3. Suppose X and Y are jointly Gaussian random variables with

$$E[X] = \mu_X,\ E[Y] = \mu_Y,\ Var(X) = \sigma_X^2,\ Var(Y) = \sigma_Y^2,\ \rho(X,Y) = \rho.$$

(a) What is the distribution of the random variable $X - Y$? Explain. (2)

(b) Compute $P\{|X| > |Y|\}$, if $\mu_X = \mu_Y,\ \sigma_X^2 = \sigma_Y^2,\ \rho = 0$. (2)

Problem 4. The input of a binary communication system is 1 with probability p and 0 with probability $1 - p$, and the output Y is a Gaussian random variable with pdf

$$f_1(x) \sim N(1,\, 1) \quad \text{when the input is 1, and} \quad f_2(x) \sim N(0,\, 1) \quad \text{when the input is 0.}$$

The receiver uses the following decision rule: If $Y > T = \frac{1}{2} + \ln \frac{1-p}{p}$, decide the input was 1; otherwise, decide the input was 0.

(a) Find $P\{\text{input } 1 \,|\, Y > T\}$ and compute this probability for $p = 0.5$. (2)

(b) Write a formula for the probability of error for the above decision rule and compute this probability for $p = 0.1$. (2)

Problem 5. $N(t)$ is a Poisson process of rate λ. Put $S_0 = 0$ and for $k \geq 1$, let S_k and T_k denote the arrival and the inter-arrival times of the process, respectively.

(a) Is $N(t)$ stationary? Prove your answer. (2)

(b) Compute the conditional probability $P\{S_1 > 1,\, S_2 > 2 \,|\, N(3) = 3\}$. (2)

(c) Suppose that t is not a point at which an event occurs, and let W_t be the random variable representing the time between t and the next occurrence of an event. How is W_t distributed? Give a proof. (4)

Problem 6. A moving average process X_n is produced as follows:

$$X_n = W_n + \alpha_1 W_{n-1} + \alpha_2 W_{n-2} + \ldots \alpha_p W_{n-p},$$

where W_n is a zero-mean white noise process with average power $\sigma_W^2 = 1$.

(a) Find $R_X(k)$ by computing $E[X_{n+k}X_n]$. (2)

(b) Find $S_X(f)$ and the transfer function $H(f)$ of the linear system that produces the moving average process. (2)

(c) Let $p = 2$ and assume that X_n has been observed at the points $n = 2$ and $n = 3$. Derive the equations which determine a and b in the best linear predictor $\hat{X}_4 = aX_2 + bX_3$ of X_4. Explain. (2)

(d) Derive a formula for computing the prediction error above. Explain. (2)