Solution to Homework 2 in TMS115 Probability and Stochastic Processes, Q1, 2004/2005

Problem 1.

a)

$$
\begin{aligned}
&P\{X(1) > 1,\ X(2) > 2,\ X(3) > 3\} \\
&= P\{X(1) > 1,\ X(2) > 2\} - P\{X(1) > 1, X(2) > 2,\ X(3) \le 3\} \\
&= P\{X(1) > 1\} - P\{X(1) > 1,\ X(2) \le 2\} - P\{X(1) > 1,\ X(2) > 2,\ X(3) \le 3\} \\
&= 1 - P\{X(1) = 0\} - P\{X(1) = 1\} \\
&- P\{X(1) = 2, X(2) - X(1) = 0\} \\
&- P\{X(1) = 2, X(2) - X(1) = 1, X(3) - X(1) = 0\} - P\{X(1) = 3, X(3) - X(1) = 0\} \\
&= 1 - e^{-\lambda} - \lambda e^{-\lambda} - \frac{\lambda^2 e^{-\lambda}}{2} \cdot e^{-\lambda} - \frac{\lambda^2 e^{-\lambda}}{2} \cdot \lambda e^{-\lambda} \cdot e^{-\lambda} - \frac{\lambda^3}{3!} e^{-\lambda} \cdot e^{-2\lambda} \\
&= \boxed{1 - (1+\lambda)e^{-\lambda} - \frac{\lambda^2}{2} e^{-2\lambda} - \frac{2\lambda^3}{3} e^{-3\lambda}}.
\end{aligned}
$$

b) Put $x_0 = 0, \quad t_0 = 0$. Let $0 \le x_1 \le \ldots \le x_n$ be integer numbers.

$$
\begin{aligned}
&P_{X(t_1), X(t_2), \ldots, X(t_n)}(x_1, \ldots, x_n) \\
&= \prod_{i=1}^{n} P\{X(t_i) - X(t_{i-1}) = x_i - x_{i-1}\} = \prod_{i=1}^{n} \frac{[\lambda(t_i - t_{i-1})]^{x_i - x_{i-1}}}{(x_i - x_{i-1})!} e^{-\lambda(t_i - t_{i-1})} \\
&= \boxed{\lambda^{x_n} e^{-\lambda t_n} \prod_{i=1}^{n} \frac{(t_i - t_{i-1})^{x_i - x_{i-1}}}{(x_i - x_{i-1})!}}.
\end{aligned}
$$

c)

$$
\begin{aligned}
C_Y(t_1, t_2) &= E\big[\big(Y(t_1) - m_Y(t_1)\big)\big(Y(t_2) - m_Y(t_2)\big)\big] \\
&= E\big[e^{-t_1}\big(X(e^{2t_1}) - m_X(e^{2t_1})\big)e^{-t_2}\big(X(e^{2t_2}) - m_X(e^{2t_2})\big)\big] \\
&= e^{-(t_1+t_2)} C_Y(e^{2t_1}, e^{2t_2}) = e^{-(t_1+t_2)} \cdot \lambda \min\{e^{2t_1},\ e^{2t_2}\} \\
&= \lambda \exp\{2\min(t_1, t_2) - t_1 - t_2\} = \boxed{\lambda e^{|t_2 - t_1|}}.
\end{aligned}
$$

Problem 2.

a) $P\{X(s) \geq X(t)\} = P\{X(s) - X(t) \geq 0\} = P\{X(t-s) - X(0) \geq 0\}$.

- If $C_X(0) \neq C_X(t-s)$, then $X(t-s) - X(0) \sim \mathcal{N}\big(0,\ 2(C_X(0) - C_X(t-s)\big)$ and $P\{X(t-s) - X(0) \geq 0\} = \frac{1}{2}$.
- If $C_X(0) = C_X(t-s)$, then $P\{X(t-s) - X(0) = 0\} = 1$ and $P\{X(t-s) - X(0) \geq 0\} = P\{X(t-s) - X(0) = 0\} = 1$.

b) For brevity, denote

$X(t_i) = X_i, \quad i = 1, 2; \quad C_X(0) = \sigma^2, \quad \rho_{X_1, X_2} = \rho$.

The random vector

$$\mathbb{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

is Gaussian with

$$m_{\mathbb{X}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad R_{\mathbb{X}} = \begin{bmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{bmatrix}.$$

We have

$$\begin{aligned} E[X_1^2 X_2^2] &= E[(X_1 - \rho\, X_2 + \rho X_2)^2 X_2^2] \\ &= E[(X_1 - \rho X_2)^2 X_2^2] + 2\rho E[(X_1 - \rho X_2) X_2^3] + \rho^2 E[X_2^4]. \end{aligned}$$

The random variables $X_1 - \rho X_2$ and X_2 are independent since they are jointly Gaussian and

$$\mathrm{Cov}(X_1 - \rho X_2, X_2) = \mathrm{Cov}(X_1, X_2) - \rho \mathrm{Cov}(X_2, X_2) = \rho\sigma^2 - \rho\sigma^2 = 0.$$

Thus

$$E[(X_1 - \rho X_2) X_2^3] = E[X_1 - \rho X_2] E[X_2^3] = 0$$

and then

$$E[X_1^2 X_2^2] = E[(X_1 - \rho X_2)^2] E[X_2^2] + \rho^2 E[X_2^4]. \tag{1}$$

We compute first

$$\begin{aligned} &E[(X_1 - \rho X_2)^2] = \mathrm{Var}(X_1 - \rho X_2) = \mathrm{Cov}(X_1 - \rho X_2, X_1 - \rho X_2) \\ &= \sigma^2 - \rho^2\sigma^2 - \rho^2\sigma^2 + \rho^2\sigma^2 = (1 - \rho^2)\sigma^2. \end{aligned} \tag{2}$$

Since $X_2 \sim \mathcal{N}(0, \sigma^2)$,

$$E[X_2^{2n}] = (2n-1)(2n-3) \ldots 1 \cdot \sigma^{2n}, \quad n = 1, 2, \ldots . \tag{3}$$

Substituting (2) and (3) in (1), we obtain

$$E[X_1^2 X_2^2] = \boxed{\sigma^4(1+2\rho^2)}.$$

Problem 3. a) Denote $X_i = X(t_i)$, $i = 1, 2, 3$, and $m_i = E[X_i]$. Since the process is Markovian, we have

$$\begin{aligned} f_{X_1,X_3}(x_1,x_3|x_2) &= \frac{f(x_1,x_2,x_3)}{f_{X_2}(x_2)} \\ &= \frac{f_{X_3}(x_3|x_1,x_2)f_{X_1,X_2}(x_1,x_2)}{f_{X_2}(x_2)} \\ &= f_{X_3}(x_3|x_2)f_{X_1}(x_1|x_2), \end{aligned} \tag{4}$$

i.e., given $X_2 = x_2$, $X_1|X_2 = x_2$ and $X_3|X_2 = x_2$ are independent. It follows from (4) that

$$\begin{aligned} &E[(X_1 - m_1)(X_3 - m_1)|X_2 = x_2] \\ &= E[(X_1 - m_1)|X_2 = x_2]E[(X_3 - m_3)|X_2 = x_2] \\ &= \frac{\rho_{X_1,X_2}\cdot\sigma_1}{\sigma_2}(x_2 - m_2)\cdot\frac{\rho_{X_2,X_3}\sigma_3}{\sigma_2}(x_2 - m_2), \end{aligned} \tag{5}$$

where in the last line of (5) we have used the fact that the random variables X_1 and X_2 are jointly Gaussian, as well as X_2 and X_3. From (5),

$$\begin{aligned} E[(X_1 - m_1)(X_2 - m_2)|X_2] &= \frac{\rho_{X_1,X_2}\sigma_1\rho_{X_2,X_3}\sigma_3}{\sigma_2^2}(X_2 - m_2)^2 \\ &= \frac{\mathrm{Cov}(X_1,X_2)\mathrm{Cov}(X_2,X_3)}{\sigma_2^4}(X_2 - m_2)^2, \end{aligned}$$

and by taking expectation from both sides we obtain

$$\mathrm{Cov}(X_1,X_2) = \frac{\mathrm{Cov}(X_1,X_2)\mathrm{Cov}(X_2,X_3)}{\mathrm{Cov}(X_2,X_2)},$$

as desired.

b) Since the autocovariance of the Wiener process is

$$C_W(t,s) = \sigma^2 \min(t,s),$$

we have $C_X(t_3,t_1) = \sigma^2 t_1$, and

$$\frac{C_X(t_3,t_2)C_X(t_2,t_1)}{C_X(t_2,t_2)} = \frac{\sigma^2 t_2\cdot\sigma^2 t_1}{\sigma^2 t_2} = \sigma^2 t_1.$$

Hence the Wiener process is Gauss-Markov.