Solution to the exam in TMS115 Probability and Stochastic Processes 2004-10-22

Problem 1. Y_{2n} ~Binomial $(2n, p)$

$$P\{Y_{2n} \le n\} P\left\{\frac{Y_{2n} - 2np}{\sqrt{2np(1-p)}} \le \frac{n - 2np}{\sqrt{2np(1-p)}}\right\}$$
$$\sim \Phi\Big(\frac{n(1-2p)}{\sqrt{2np(1-p)}}\Big) = \Phi\Big(\frac{\sqrt{n}(1-2p)}{\sqrt{2p(1-p)}}\Big).$$

Then

$$\lim_{n \longrightarrow \infty} P\{Y_{2n} \le n\} = \begin{cases} \frac{1}{2}, & p = \frac{1}{2}, \\ 1, & p < \frac{1}{2}, \\ 0, & p > \frac{1}{2}. \end{cases}$$

Problem 2. The power of the exponent in the joint pdf can be written as

$$\frac{1}{2(1-\rho^2)}\left[\frac{z_1^2}{\frac{1}{2(1-\rho^2)}} - \frac{2}{\sqrt{2}}\frac{z_1}{\frac{1}{\sqrt{2(1-\rho^2)}}} \cdot \frac{z_2}{\frac{1}{\sqrt{2(1-\rho^2)}}} + \frac{z_2^2}{\frac{1}{2(1-\rho^2)}}\right].$$

Thus

$$\rho\,(Z_1,\, Z_2) = \rho = \frac{1}{\sqrt{2}}, \quad E\,[Z_i] = 0, \quad Var\,(Z_i) = \sigma_i^2 = \frac{1}{2(1-\rho^2)} = 1, \quad i = 1,\, 2.$$

a) $\mathrm{Cov}\,(Z_1 - Z_2/\sqrt{2},\, Z_2) = \mathrm{Cov}\,(Z_1,\, Z_2) - \frac{1}{\sqrt{2}}\mathrm{Cov}\,(Z_2,\, Z_2)$
$= \rho\,\sigma_1\sigma_2 - \frac{1}{\sqrt{2}}\sigma_2^2 = 0.$

b) The random variables $Z_1 - Z_2/\sqrt{2}$ and Z_2 are jointly Gaussian and uncorrelated. Hence they are independent and we thus have
$E\,[Z_1^2 Z_2] = E\left[\big(Z_1 - Z_2/\sqrt{2} + Z_2/\sqrt{2}\big)^2 Z_2\right]$
$= E\left[\big(Z_1 - Z_2/\sqrt{2}\big)^2 Z_2\right] - 2E\left[\big(Z_1 - Z_2/\sqrt{2}\big) Z_2^2/\sqrt{2}\right] + E\left[Z_2^3/2\right]$
$= E\left[\big(Z_1 - Z_2/\sqrt{2}\big)^2\right] E\,[Z_2] - 2E\left[Z_1 - Z_2/\sqrt{2}\right] E\left[Z_2^2/\sqrt{2}\right] + E\left[Z_2^3/2\right] = 0,$
since
$E\,[Z_2] = E\left[Z_1 - Z_2/\sqrt{2}\right] = E\left[Z_2^3/2\right] = 0.$

Problem 3. $X \sim$ Binomial $(M,\, p)$

a) $Y=$ the number of discarded voices

$$Y = \begin{cases} 0, & \text{if } \ X \le N, \\ X - N & \text{if } \ X > N. \end{cases}$$

$$E[Y] = \sum_{j=1}^{M-N} j P\{X = N + j\} = \sum_{j=1}^{M-N} j \binom{M}{N+j} p^{N+j} (1-p)^{M-N-j}.$$

b) $P\{Y = 0\} = P\{X \le N\} \approx \Phi\Big(\dfrac{N - M \cdot p}{\sqrt{Mp(1-p)}}\Big)$

$= \Phi\Big(\dfrac{16 - 45 \cdot 1/3}{\sqrt{45 \cdot 1/3 \cdot 2/3}}\Big) = 1 - Q(0.31) = 0.62$

Problem 4. Let $N_1(t)$ and $N_2(t)$ be the two Poisson processes on line 1 and line 2, and T_1 and T_2 be the waiting times for the first event in $N_1(t)$ and $N_2(t)$, respectively.

a) The random variables T_1 and T_2 are independent and exponentially distributed of rates λ and μ, correspondingly. We have

$$\begin{aligned} P\{T_1 < T_2\} &= \int_0^\infty P\{T_1 < T_2 | T_2 = x\} f_{T_2}(x) dx \\ &= \int_0^\infty P\{T_1 < x | T_2 = x\} f_{T_2}(x) dx = \int_0^\infty P\{T_1 < x\} f_{T_2}(x) \\ &= \int_0^\infty (1 - e^{-\lambda x})\, \mu\, e^{-\mu x} dx = \frac{\lambda}{\lambda + \mu}. \end{aligned}$$

b) The waiting time for the first message to come is $T = \min\,(T_1,\, T_2)$.

$$\begin{aligned} &P\{\min\,(T_1,\, T_2) \le t\} = 1 - P\{\min\,(T_1,\, T_2) > t\} \\ &= 1 - P\{T_1 > t,\, T_2 > t\} = 1 - P\{T_1 > t\}\, P\{T_2 > t\} = 1 - e^{-(\lambda+\mu)\, t}. \end{aligned}$$

Then T is an exponential random variable with

$$f_T(t) = (\lambda + \mu) e^{-(\lambda+\mu)\, t}.$$

c) Let $N(t) = N_1(t) + N_2(t)$. Since $N_1(t)$ and $N_2(t)$ are independent, $N(t)$ is a Poisson process of rate 2λ. We have

$$\begin{aligned}
&P\{N(t/2) \geq 1,\, N(t) - N(t/2) \geq 1 | N(t) = 3\} \\
&= \frac{P\{N(t/2) \geq 1,\, N(t) - N(t/2) \geq 1,\, N(t) = 3\}}{P\{N(t) = 3\}} \\
&= \frac{2 \cdot P\{N(t/2) = 1\}\, P\{N(t/2) = 2\}}{P\{N(t) = 3\}} = \frac{2 \cdot \frac{\lambda t}{2} e^{-\frac{\lambda t}{2}} \cdot \frac{(\lambda t)^2}{8} e^{-\frac{\lambda t}{2}}}{\frac{(\lambda t)^3}{6} e^{-\lambda t}} = \frac{3}{4}.
\end{aligned}$$

Problem 5.

$$\begin{aligned}
&m_Z(t) = 0, \\
&E[Z(t+\tau)Z(t)] = E\big[\big(bX(t+\tau) - Y(t+\tau-b)\big)\big(bX(t) - Y(t-b)\big)\big] \\
&= b^2 R_X(\tau) - bR_{X,Y}(\tau + b) - bR_{Y,X}(\tau - b) + R_Y(\tau).
\end{aligned}$$

Hence $Z(t)$ is WSS.

Problem 6.

a) $R_{Y,X}(n+k,\, n) = E[(X_{n+k} + \beta X_{n+k-1})X_n] = R_X(k) + \beta R_X(k-1)$

$S_{Y,X}(f) = (1 + \beta e^{-j2\pi f})S_X(f)$

$$S_X(f) = \sigma^2 \sum_{k=-\infty}^{\infty} \alpha^{|k|} e^{-j2\pi f k} = \frac{\sigma^2(1-\alpha^2)}{1 + \alpha^2 - 2\alpha\cos 2\pi f}$$

b) The unit impuls response is

$$h_n = \begin{cases} 1, & n = 0, \\ \beta, & n = 1, \\ 0, & \text{otherwise,} \end{cases}$$

and the transfer function is $H(f) = 1 + \beta e^{-j2\pi f}$.

Thus

$$\begin{aligned}
S_Y(f) &= |1 + \beta e^{-j2\pi f}|^2 S_X(f) = \sigma^2(1-\alpha^2) \cdot \frac{1 + \beta^2 + 2\beta\cos 2\pi f}{1 + \alpha^2 - 2\alpha\cos 2\pi f} \\
&= \sigma^2(1-\alpha^2) \cdot \frac{\beta}{\alpha} \cdot \frac{\frac{1+\beta^2}{2\beta} + \cos 2\pi f}{\frac{1+\alpha^2}{2\alpha} - \cos 2\pi f} \\
&= \sigma^2(1-\alpha^2) \cdot \frac{\beta}{\alpha} \cdot \Big[-1 + \frac{\frac{1+\beta^2}{2\beta} + \frac{1+\alpha^2}{2\alpha}}{\frac{1+\alpha^2}{2\alpha} - \cos 2\pi f} \Big].
\end{aligned}$$

$S_Y(f)$ is constant in f if and only if

$$\frac{1+\beta^2}{2\beta} = -\frac{1+\alpha^2}{2\alpha}$$
$$\Updownarrow$$
$$\alpha(1+\beta^2)+\beta(1+\alpha^2)=0$$
$$\Updownarrow$$
$$(\alpha+\beta)(1+\alpha\beta)=0$$

Y_n is a white-noise process if and only if either $\beta=-\alpha$ or $\beta=-1/\alpha$.

Problem 7.

a) We have $P(D)Y=W$, where $P(D)=D^0-\frac{1}{2}D$ is the polynomial shift operator. Its characteristic polynomial is $P(z)=1-\frac{1}{2}z, \quad |z|>1$, and

$$P(z)^{-1}=\frac{1}{1-\frac{1}{2}z}=\sum_{k=0}^{\infty}2^{-k}z^k.$$

Thus

$$P(D)^{-1}=\sum_{k=0}^{\infty}2^{-k}D^k \quad \text{and} \quad Y_n=\sum_{k=0}^{\infty}2^{-k}W_{n-k}\,.$$

The unit-impulse response of the system producing Y_n is

$$h_n=\begin{cases}2^{-n}, & n\geq 0,\\ 0, & n<0.\end{cases}$$

For $k\geq 0$, we have

$$\begin{aligned}R_Y(k)&=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}2^{-r}\cdot 2^{-s}R_W(k+r-s)\\ &=\sum_{r=0}^{\infty}2^{-r}\cdot 2^{-k-r}\sigma_W^2=\sigma_W^2\,2^{-k}\sum_{r=0}^{\infty}2^{-2r}=\frac{4}{3}\sigma_W^2\,2^{-k}.\end{aligned}$$

Then

$$R_Y(k)=\frac{4}{3}\sigma_W^2\left(\frac{1}{2}\right)^{|k|}, \quad k=0,\pm 1,\ldots.$$

b) $\hat{Y}_n=h_2Y_{n-2}+h_3Y_{n-3}$ - the estimation. The orthogonality condition

$$Y_n-\hat{Y}_n\perp Y_{n-2} \quad \text{and} \quad Y_n-\hat{Y}_n\perp Y_{n-3}$$

implies

$$h_2 R_X(0) + h_3 R_X(1) = R_X(2)$$
$$h_2 R_X(1) + h_3 R_X(0) = R_X(3)$$

or

$$4h_2 + 2h_3 = 1$$
$$2h_2 + 4h_3 = 1/2.$$

Thus $h_2 = \dfrac{1}{4}, \quad h_3 = 0, \quad$ and

$$E[e_n^2] = E[(Y_n - \hat{Y}_n)e_n] = E[Y_n(Y_n - \hat{Y}_n)] = R_X(0) - h_2 R_X(2) = 4 - \frac{1}{4} = 3.75$$