# Solution to homework 1 in TMS115

# Probability and Stochastic Processes, 2005/2006

• There are 10 total points in the homework. One needs 7.5 points for a bonus of 2 points out of 30 in the written examination.

**Problem 1.** Denote

$$\begin{aligned} R &= \{\text{Getting a question righ}\} \\ TN &= \{\text{You think you know the answer}\} \\ G &= \{\text{You guess the answer}\}. \end{aligned}$$

(a)

$$P(R) = P(R|TN)P(TN) + P(R|G)P(G) = 0.8 \times 0.75 + 0.2 \times 0.25 = 0.65.$$

(b)

$$P(G|R) = P(R|G)P(G)/P(R) = 1/13 \approx 0.07.$$

**Problem 2.** Let $X$ be the random variable denoting the number of hosts attempting to access the channel during one time slot and $p_C$ be the probability for a collision during one time slot. We have $X \sim Bin\,(n,\,p)$ and

$$p_C = P\{X \geq 2\} = 1 - P\{X \leq 1\} = 1 - (1-p)^n - np(1-p)^{n-1}.$$

(a) Assume there are k slots. The number $Y$ of the slots wasted due to collisions is a $Bin(k, p_C)$ random variable. Since $E[Y] = kp_C$ the fraction of the wasted slots is $p_C$.

(b) Since $np = 5 + 1/n \to 5$ when $n \to \infty$, $X$ is well approximated by a $Poisson(5)$ random variable. Thus $p_C \approx 1 - e^{-5} - 5e^{-5} = 0.96$.

**Problem 3.**

(a)

$$P\{signal\ present\,|X = k\}$$

$$= \frac{P\{X = k|signal\ present\}p}{P\{X = k|signal\ present\}p + P\{X = k|signal\ absent\}(1-p)}$$

$$= \frac{\lambda_1^k\, e^{-\lambda_1}\, p}{\lambda_1^k\, e^{-\lambda_1}\, p + \lambda_0^k\, e^{-\lambda_0}\,(1-p)}$$

and

$$P\{signal\ absent\,|X = k\} = \frac{\lambda_0^k\, e^{-\lambda_0}\,(1-p)}{\lambda_1^k\, e^{-\lambda_1}\, p + \lambda_0^k\, e^{-\lambda_0}\,(1-p)}.$$

(b) Since $\lambda_1 > \lambda_0$ the inequality

$$P\{signal\ present\,|X=k\} > P\{signal\ absent\,|X=k\}$$

is equivalent to

$$k > \frac{\ln\frac{1-p}{p} + \lambda_1 - \lambda_0}{\ln\frac{\lambda_1}{\lambda_0}}$$

and the threshold $T$ then is

$$T = \frac{\ln\frac{1-p}{p} + \lambda_1 - \lambda_0}{\ln\frac{\lambda_1}{\lambda_0}}.$$

(c) Denote by $p_E$ the probability of error. We have

$$p_E = P\{X < T|signal\ present\}\,p + P\{X > T|signal\ absent\}\,(1-p)$$

$$= p\sum_{k=0}^{\lfloor T\rfloor}\frac{\lambda_1^k\,e^{-\lambda_1}}{k!} + (1-p)\sum_{k=\lceil T\rceil}^{\infty}\frac{\lambda_1^k\,e^{-\lambda_1}}{k!}.$$

**Problem 4.**

(a)

$$f_{X,R}(x,r) = f_X(x|r)f_R(r) = \frac{\lambda^\alpha}{\Gamma(\alpha)}\,r^\alpha\,e^{-\lambda\,(x+r)}.$$

(b)

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,R}(x,r)dr = \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_{-\infty}^{\infty} r^\alpha\,e^{-(\lambda+x)\,r}dr$$
$$= \frac{\lambda^\alpha}{\Gamma(\alpha)}\cdot\frac{\Gamma(\alpha+1)}{(\lambda+x)^{\alpha+1}} = \frac{\alpha\,\lambda^\alpha}{(\lambda+x)^{\alpha+1}}, \quad x > 0.$$

(c)

$$E[X] = E[E[X|Y] = E[1/R] = \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty r^{\alpha-2}\,e^{-\lambda\,r}dr$$
$$= \frac{\lambda^\alpha}{\Gamma(\alpha)}\cdot\frac{\Gamma(\alpha-1)}{\lambda^{\alpha-1}} = \frac{\lambda}{\alpha-1}, \quad \alpha > 1.$$

$$Var(X) = Var(E[X|Y]) + E[Var(X|Y)] = Var(1/R) + E[1/R^2]$$
$$= 2E[1/R^2] - \big(E[1/R]\big)^2 = 2\,\frac{\lambda^\alpha}{\Gamma(\alpha)}\cdot\frac{\Gamma(\alpha-2)}{\lambda^{\alpha-2}} - \frac{\lambda^2}{(\alpha-1)^2}$$
$$= \frac{2\lambda^2}{(\alpha-1)(\alpha-2)} - \frac{\lambda^2}{(\alpha-1)^2} = \frac{\lambda^2\,\alpha}{(\alpha-1)^2(\alpha-2)}, \quad \alpha > 2.$$