Solution to homework 2 in TMS115

Probability and Stochastic Processes, 2005/2006

• There are 10 total points in the homework. One needs 7.5 points for a bonus of 2 points out of 30 in the written examination.

Problem 1. Denote

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}, \quad m_{\mathbf{X}} = \begin{bmatrix} E[X_1] \\ E[X_2] \\ E[X_3] \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}, \quad m_{\mathbf{Y}} = \begin{bmatrix} E[Y_1] \\ E[Y_2] \\ E[Y_3] \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

The matrix A is non-degenerate, thus $\mathbf{X} = A^{-1}\mathbf{Y}, \quad m_{\mathbf{X}} = A^{-1}\, m_{\mathbf{Y}}$. Let $R_{\mathbf{X}}$ be the covariance matrix of X. We have

$$\begin{aligned} f_{\mathbf{Y}}(\mathbf{y}) &= f_{\mathbf{X}}(A^{-1}\mathbf{y})\, |\det A^{-1}| \\ &= \frac{|\det A^{-1}|}{(2\pi)^{3/2}\sqrt{\det R_{\mathbf{X}}}} \exp\Big\{ -\frac{1}{2}\big(A^{-1}(\mathbf{y}-m_{\mathbf{Y}})\big)^T R_{\mathbf{X}}^{-1} \big(A^{-1}(\mathbf{y}-m_{\mathbf{Y}})\big)\Big\} \\ &= \frac{|\det A^{-1}|}{(2\pi)^{3/2}\sqrt{\det R_{\mathbf{X}}}} \exp\Big\{ -\frac{1}{2}(\mathbf{y}-m_{\mathbf{Y}})^T (A\, R_{\mathbf{X}}\, A^T)^{-1} (\mathbf{y}-m_{\mathbf{Y}})\Big\} \\ &= \frac{1}{(2\pi)^{3/2}\sqrt{\det R}} \exp\Big\{ -\frac{1}{2}(\mathbf{y}-m_{\mathbf{Y}})^T R^{-1} (\mathbf{y}-m_{\mathbf{Y}})\Big\}, \end{aligned}$$

where $R = A\, R_{\mathbf{X}}\, A^T$. Hence $\mathbf{Y} \sim N(m_{\mathbf{Y}},\ R)$.

Problem 2.

(a)

$$\begin{aligned} P\{X_1(t) = k\} &= \sum_{n=k}^{\infty} P\{X_1(t) = k|\, N(t) = n\} P\{N(t) = n\} \\ &= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \frac{(\lambda\, t)^n}{n!} e^{-\lambda t} \\ &= \frac{(\lambda\, p\, t)^k}{k!} e^{-\lambda t} \sum_{m\,=\,n-k=0}^{\infty} \frac{(\lambda\, t\,(1-p))^m}{m!} = \frac{(\lambda\, p\, t)^k}{k!} e^{-\lambda\, p\, t}. \end{aligned}$$

Thus $X_1(t) \sim \text{Poisson}(\lambda pt)$. Since the process has independent and stationary increments and $N_1(0) = 0$, it is a Poisson process of rate λp. Similarly, $X_2(t)$ is a Poisson process of rate $\lambda(1-p)$. (2)

(b)

$$\begin{aligned} P\{X_1(t) = k,\, X_2(t) = m\} &= P\{X_1(t) = k,\, X_2(t) = m \,|\, X(t) = k+m\} P\{X(t) = k+m\} \\ &= \binom{(k+m)}{k} p^k (1-p)^m \frac{(\lambda t)^{k+m}}{(k+m)!}\, e^{-\lambda t} = \frac{(\lambda p t)^k}{k!}\, e^{-\lambda p t} \cdot \frac{(\lambda(1-p)t)^m}{m!}\, e^{-\lambda(1-p)t} \\ &= P\{X_1(t) = k\} P\{X_2(t) = m\}. \end{aligned}$$

Thus $X_1(t)$ and $X_2(t)$ are independent. (2)

Problem 3. Denote $p_e(t) = P\{X(t) \text{ is even }\}$. We know (the book, p. 350)

$$p_e(t) = \frac{1 + e^{-2\lambda t}}{2}.$$

(a) $t > 0: \quad P\{Y(t) = 1\} = p_e(t), \quad P\{Y(t) = -1\} = 1 - p_e(t), \quad E[Y(t)] = e^{-2\lambda t}.$
$m_Y(0) = 1 \quad \text{and} \quad m_Y(t) = e^{-2\lambda t}, \quad t > 0.$

$t \geq 0, \quad \tau \geq 0:$
$R_Y(t,\ t + \tau) = E[Y(t)Y(t + \tau)] = 1 \cdot p_e(\tau) + (-1) \cdot (1 - p_e(\tau)) = e^{-2\lambda\tau}.$

$Y(t)$ is not WSS, since $m_Y(t)$ is not a constant. The process is known as the *semirandom telegraph signal* because its initial value $Y(0)$ is not random. (2)

(b) $Z(t)$ is the random telegraph signal which is WSS, see the book, p. 350. (1)

(c) $S_Y(f) = \dfrac{4\lambda}{4\lambda + 4\pi^2 f^2}$, see the book, p. 405. (1)