Solution to the exam in TMS115 Probability and Stochastic Processes 2005-01-14

Problem 1. Let X be the input of the channel and Y be the output.

a)

$$P\{Y=1\} = \frac{1}{2}(1-\varepsilon_2) + \frac{1}{2}\varepsilon_1 = \frac{1}{2}(1+\varepsilon_1-\varepsilon_2)$$

$$P\{X=1|Y=1\} = \frac{\Pr\{X=1,Y=1\}}{\Pr\{Y=1\}} = \frac{\frac{1}{2}(1-\varepsilon_2)}{\frac{1}{2}(1+\varepsilon_1-\varepsilon_2)} = \frac{1-\varepsilon_2}{1+\varepsilon_1-\varepsilon_2}$$

$$P\{X=0|Y=1\} = \frac{\varepsilon_1}{1+\varepsilon_1-\varepsilon_2}$$

From $\varepsilon_i < 1/2$ we get $P\{X=1|Y=1\} > P\{X=0|Y=1\}$. Thus the most probable value of $X=1$ when $Y=1$ is 1.

b) Since X and Y are binary random variables

$$P\{X=1,Y=1\} = \Pr\{X=1\}\Pr\{Y=1\} \Longleftrightarrow X,Y-independent$$

$$\begin{aligned}
p(1-\varepsilon) = p[p(1-\varepsilon)+(1-p)\varepsilon] &= p[p+\varepsilon(1-2p)\\
1-\varepsilon &= p+\varepsilon(1-2p)\\
1-p = \varepsilon(2-2p) &= 2\varepsilon(1-p)\\
\varepsilon &= 1/2
\end{aligned}$$

Problem 2. Denote by $S_1, S_2, \ldots$ the arrival times of the Poisson process. By the CLT,

$$P\{S_{601} \le 60\} \approx \Phi\left(\frac{60-601.\frac{1}{10}}{\sqrt{601.\frac{1}{100}}}\right) = \Phi(-0.04) = 0.48.$$

Problem 3.

a)

$$\mathbf{E}[(X-Y)^2] = \mathbf{E}[X^2] + \mathbf{E}[Y^2] - 2\mathbf{E}[XY] = 2\left[\mathbf{E}[X^2] - (\mathbf{E}[X])^2\right] = 2\mathbf{Var}(X) = \frac{1}{6}$$

b) The range of Z is $(0, 2)$. When $0 < z < 1$, we have

$$P\{X+Y<z\} = P\{Y<z-X\} = \int_0^z P\{Y<z-x\}\,dx = \frac{z^2}{2},$$

and when $1 \le z < 2$,

$$\begin{aligned} P\{X+Y<z\} = P\{Y<z-X\} &= \int_0^1 P\{Y<z-x\}\,dx \\ &= \int_0^{z-1} P\{Y<z-x\}\,dx + \int_{z-1}^1 P\{Y<z-x\}\,dx \\ &= z-1+z-\frac{z^2}{2} = 2z-\frac{z^2}{2}-1. \end{aligned}$$

Thus

$$f_Z(z) = z \quad \text{when} \quad 0<z<1 \quad \text{and} \quad f_Z(z) = 2-z \quad \text{when} \quad 1 \le z < 2.$$

$$E[Z|Z>1] = \int_1^2 z f_Z(z|Z>1)\,dz = \frac{1}{1-F_Z(1)}\int_1^2 z(2-z)\,dz = \frac{4}{3}.$$

Problem 4.

a) Assume $n \ge m$:

$$p(m,n) = \Pr\{N(1)=m, N(4)=n\} = \Pr\{\underbrace{N(1)}_{Poisson(\lambda)} = m\}\Pr\{\underbrace{N(4)-N(1)}_{Poisson(3\lambda)} = n-m\} =$$

$$= \frac{\lambda^m}{m!}e^{-\lambda}\frac{(3\lambda)^{(n-m)}}{(n-m)!}e^{-3\lambda} = \frac{3^{n-m}\lambda^n e^{-4\lambda}}{m!(n-m)!}$$

b)

$$\Pr\{N(1)\ge 2, N(4)\le 3\} = p(2,2)+p(3,3)+p(2,3) =$$

$$= \left[\frac{3^0\lambda^2}{2!}+\frac{3^0\lambda^3}{3!}+\frac{3^1\lambda^3}{2!}\right]e^{-4\lambda} = \left[\frac{1}{2}\lambda^2+\frac{5}{3}\lambda^3\right]e^{-4\lambda}$$

c)

$$C_X(t_1,t_2) = e^{-(t_1+t_2)}C_N(e^{2t_1}, e^{2t_2}) = e^{-(t_1+t_2)}\lambda \min(e^{2t_1}, e^{2t_2})$$
$$= \lambda e^{-(t_1+t_2)+2\min(t_1,t_2)} = \lambda e^{-|t_1-t_2|}$$

Problem 5.

$$S_X(f) = (\mathcal{F}\{16e^{-5|\tau|}\} * \mathcal{F}\{\cos(2\pi\tau)\}(f) + (\mathcal{F}\{8\cos(10\pi\tau)\})(f)$$

$$\mathcal{F}\{e^{-\alpha|\tau|}\} = \frac{2\alpha}{\alpha^2 + 4\pi^2 f^2}$$

$$\mathcal{F}\{\cos(2\pi\alpha\tau)\} = \frac{1}{2}\left[\delta(f-\alpha) + \delta(f+\alpha)\right]$$
$$S_X(f) = 16\frac{10}{25+4\pi^2 f^2} * \frac{1}{2}\left[\delta(f-1) + \delta(f+1)\right] +$$
$$+8\frac{1}{2}\left[\delta(f-5) + \delta(f+5)\right] = 80\left[\frac{1}{25+4\pi^2(f-1)^2} + \frac{1}{25+4\pi^2(f+1)^2}\right] + 4[\delta(f-5) + \delta(f+5)]$$
$$S_X(0) = 80\left[\frac{2}{25+4\pi^2}\right] = 2.48, \quad R_X(0) = 24$$

Problem 6.

a)

$$Y(t) = \int_0^1 2\cos(2\pi(t-u)+\theta)[5\delta(u)+3]du = 10\cos(2\pi t + \theta)$$

b)

$$m_X = 0; \quad m_Y = m_X \int_0^1 h(t)dt = 0$$
$$Var(Y(t)) = E[Y^2(0)] = 100\int_0^{2\pi} \frac{1}{2\pi}\cos^2\theta\, d\theta = \frac{50}{\pi}\int_0^{2\pi} \frac{1+\cos 2\theta}{2}d\theta = \frac{25}{\pi}\left[2\pi + \frac{\sin 2\theta}{2}\Big|_0^{2\pi}\right] = 50$$

Problem 7.

a)

$$R_X(k) = 9(1/3)^{|k|}, \quad k = 0, \pm 1, \ldots$$

$$\begin{bmatrix} R_X(0) & R_X(3) \\ R_X(3) & R_X(0) \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} R_X(2) \\ R_X(1) \end{bmatrix}$$

$$\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} 9/91 \\ 30/91 \end{bmatrix} = \begin{bmatrix} 0.0989 \\ 0.3297 \end{bmatrix}$$

b)

$$e_n^2 = R_X(0) - h_1 R_X(2) - h_2 R_X(1) = 7.912$$